23 ก.พ. 60

2.9 อนุพันธ์ของฟังก์ชันตรีโกณมิติ

ทบทวน $\lim_{x\to 0} \frac{\sin x}{x} = 1$ และ $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$

Thm

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

Pf:

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \left[\sin x \cdot \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h}\right] \\
&= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} \\
&= 0 + \cos x = \cos x
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\csc x &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) \\
&= \frac{\sin x \frac{d}{dx} 1 - 1 \cdot \frac{d}{dx}\sin x}{\sin^2 x} \\
&= \frac{0 - \cos x}{\sin^2 x} \\
&= -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x
\end{aligned}$$

ถ้าให้ $u = g(x)$ เป็นฟังก์ชันที่หาอนุพันธ์
โดยกฎลูกโซ่จะได้

$$\frac{d}{dx}\sin u = \cos u \cdot \frac{du}{dx} \qquad \frac{d}{dx}\cos u = -\sin u \cdot \frac{dy}{dx}$$

$$\frac{d}{dx}\sec u = \sec u \tan u \frac{dy}{dx} \qquad \frac{d}{dx}\csc u = -\csc u \cot u \frac{du}{dx}$$

$$\frac{d}{dx}\tan u = \sec^2 u \frac{dy}{dx} \qquad \frac{d}{dx}\cot u = -\csc^2 u \frac{dy}{dx}$$

Ex

$$\frac{d}{dx}\sin(2x^3+7x) = \cos(2x^3+7x)\cdot(6x^2+7)$$

$$\frac{d}{dx}\cos(5x^4) = -\sin(5x^4)\cdot 20x^3$$

$$\frac{d}{dx}\cos(5x)^4 = -\sin(5x)^4 \cdot 4(5x)^3\cdot 5$$

$$= -20(5x)^3\sin(5x)^4$$

$$\frac{d}{dx}\cos^4(5x) = \frac{d}{dx}[\cos(5x)]^4$$

$$= 4\cos^3(5x)\cdot(-1)\sin(5x)\cdot 5$$

$$= -20\cos^3(5x)\sin(5x)$$

Ex

$$\frac{d}{dx}\tan(\sin(x^2+1)) = \sec^2(\sin(x^2+1))\,\frac{d}{dx}\sin(x^2+1)$$

$$= \sec^2(\sin(x^2+1))\cdot\cos(x^2+1)\cdot 2x$$

Ex

$$\frac{d}{dx}(\sin 3x + \csc 7x)^3 = 3(\sin 3x + \csc 7x)^2 \cdot \frac{d}{dx}(\sin 3x + \csc 7x)$$

$$= 3(\sin 3x + \csc 7x)^2(3\cos 3x - 7\csc 7x \cot 7x)$$

Ex $\frac{d}{dx}\left[\sin^3(x^2+5)\cos(8x)\right]$

$$= \sin^3(x^2+5)\frac{d}{dx}\cos(8x) + \cos(8x)\frac{d}{dx}\sin^3(x^2+5)$$

$$= -8\sin^3(x^2+5)\sin(8x) + \cos(8x)\cdot 3\sin^2(x^2+5)\cdot\cancel{\frac{d}{dx}} \cdot \frac{d}{dx}\sin(x^2+5)$$

$$= -8\sin^3(x^2+5)\sin(8x) + 3\cos(8x)\sin^2(x^2+5)\cos(x^2+5)\cdot 2x$$

Ex $\frac{d}{dx}(\sin x + \cot x^2)^{\tan x}$ $\qquad \ln a^b = b\ln a$

Sol ให้ $y = (\sin x + \cot x^2)^{\tan x}$

$$\ln y = \ln(\sin x + \cot x^2)^{\tan x}$$

$$\ln y = \tan x \ln(\sin x + \cot x^2)$$

หา $\frac{d}{dx}$ ของทั้ง 2 ข้าง

$$\frac{1}{y}\frac{dy}{dx} = \tan x\,\frac{d}{dx}(\sin x + \cot x^2) + \ln(\sin x + \cot x^2)\,\frac{d}{dx}\tan x$$

$$\frac{1}{y}\frac{dy}{dx} = \tan x \cdot \frac{\cos x - \csc^2 x^2 \cdot 2x}{\sin x + \cot x^2} + \ln(\sin x + \cot x^2)\cdot \sec^2 x$$

$$\frac{dy}{dx} = \left[\tan x \cdot \frac{\cos x - \csc^2 x^2 \cdot 2x}{\sin x + \cot x^2} + \ln(\sin x + \cot x^2)\cdot \sec^2 x\right] \cdot (\sin x + \cot x^2)^{\tan x}$$

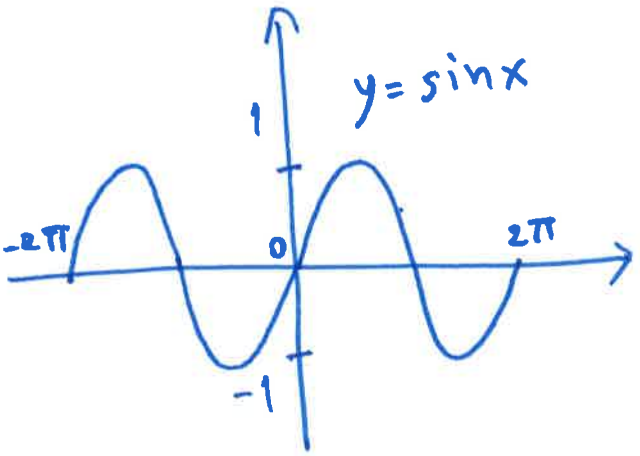
Ex $\frac{d}{dx}\pi^{\sin(3x+5)}$

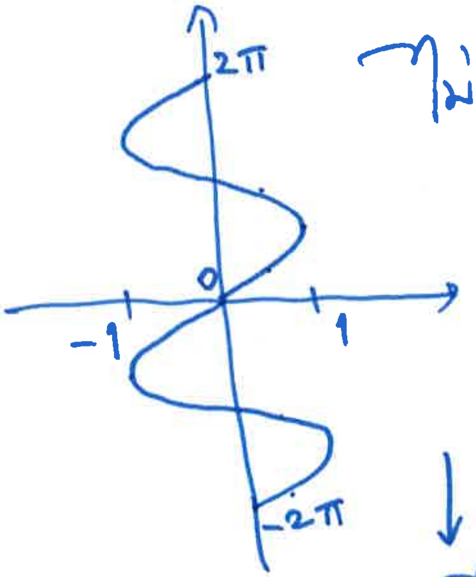
$$= \ln\pi \cdot \pi^{\sin(3x+5)} \cos(3x+5)\cdot 3$$

ทำแบบฝึก 2.9

1. คี่ 2. คู่ 3. คู่

2.10 อนุพันธ์ของฟังก์ชันตรีโกณมิติผกผัน

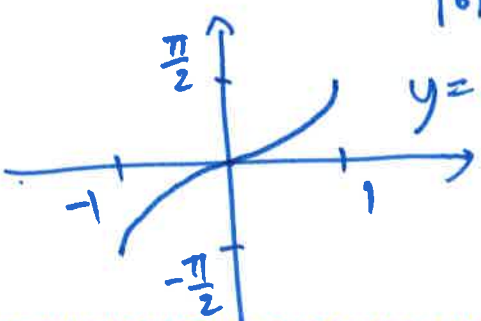




ไม่เป็นฟังก์ชัน

↓ ตัดมา

ได้ฟังก์ชัน



$y = \sin^{-1} x$

ฟังก์ชัน	โดเมน	เรนจ์
$y=\sin^{-1}x$	$[-1,1]$	$[-\frac{\pi}{2},\frac{\pi}{2}]$
$y=\cos^{-1}x$	$[-1,1]$	$[0,\pi]$
$y=\tan^{-1}x$	$(-\infty,\infty)$	$(-\frac{\pi}{2},\frac{\pi}{2})$
$y=\csc^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$(-\pi,-\frac{\pi}{2}]\cup(0,\frac{\pi}{2}]$
$y=\sec^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$[0,\frac{\pi}{2})\cup[\pi,\frac{3\pi}{2})$
$y=\cot^{-1}x$	$(-\infty,\infty)$	$(0,\pi)$

ดู หน้า 108-109

Ex จงหาค่าของ $\sin^{-1}(\frac{1}{2})$

ให้ $\sin^{-1}(\frac{1}{2}) = \theta$

$$\frac{1}{2} = \sin\theta$$

$$\therefore \underset{\sin^{-1}(\frac{1}{2})}{\underset{\parallel}{\theta}} = 30^\circ = \frac{\pi}{6}$$

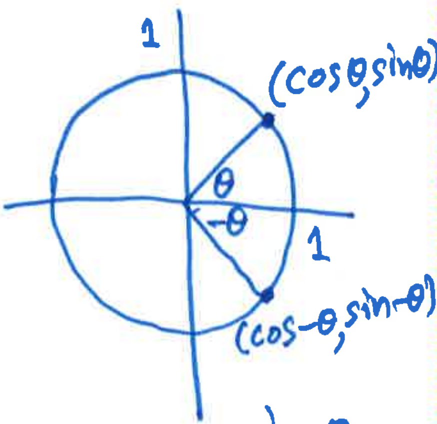
Ex $\cos^{-1}(\frac{1}{2})$ = มุม θ อะไรที่ $\cos\theta = \frac{1}{2}$

$= 60^\circ = \frac{\pi}{3}$

Ex $\sin^{-1}(-\frac{\sqrt{3}}{2})$ = มุม θ ที่ $\sin\theta = -\frac{\sqrt{3}}{2}$

= − มุม β ที่ $\sin\beta = \frac{\sqrt{3}}{2}$

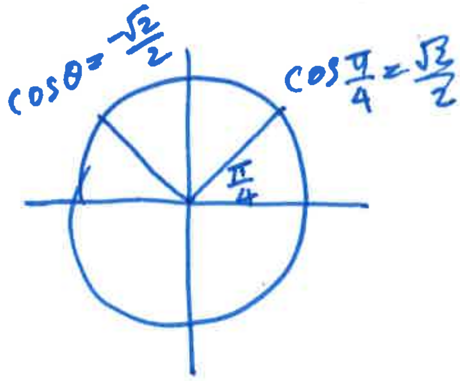
$= -\frac{\pi}{3}$



$\sin(-\theta) = -\sin\theta$

Ex $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) =$ มุม θ ที่ $\cos\theta = -\frac{\sqrt{2}}{2}$

$$= \frac{3\pi}{4}$$



Thm

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}, \quad -1<x<1 \qquad \frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}, \quad -1<x<1$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}, \quad |x|>1 \qquad \frac{d}{dx}\csc^{-1}x = -\frac{1}{x\sqrt{x^2-1}}, \quad |x|>1$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$$

Pf: ให้ $y = \sin^{-1}x$

จะได้ $\sin y = x$ โดยที่ $|x| \leq 1$ และ $|y| \leq \frac{\pi}{2}$

$$\frac{d}{dx}\sin y = \frac{d}{dx}x$$

$$\cos y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} \quad \text{เมื่อ } \cos y \neq 0$$

เนื่องจาก $\sin^2 y + \cos^2 y = 1$

จะได้ $\cos^2 y = 1 - \sin^2 y$

$$\cos y = \pm\sqrt{1-\sin^2 y}$$

แต่ $|y| \leq \frac{\pi}{2}$ จะได้ $\cos y > 0$ จะได้

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

$$\therefore \frac{d}{dx}\sin^{-1}x = \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

ถ้า $u=g(x)$ เป็นฟังก์ชันที่หาอนุพันธ์ได้ โดยกฎลูกโซ่จะได้

$$\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{dy}{dx}, \quad -1<u<1$$

$$\frac{d}{dx}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}}\frac{dy}{dx}, \quad -1<u<1$$

$$\frac{d}{dx}\sec^{-1}u = \frac{1}{u\sqrt{u^2-1}}\frac{dy}{dx}, \quad |u|>1$$

$$\frac{d}{dx}\csc^{-1}u = -\frac{1}{u\sqrt{u^2-1}}\frac{dy}{dx}, \quad |u|>1$$

$$\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{dy}{dx}$$

$$\frac{d}{dx}\cot^{-1}u = -\frac{1}{1+u^2}\frac{dy}{dx}$$

Ex

$$\frac{d}{dx}\sin^{-1}(2x^3) = \frac{1}{\sqrt{1-(2x^3)^2}}\cdot 6x^2 = \frac{6x^2}{\sqrt{1-4x^6}}$$

$$\frac{d}{dx}\cot^{-1}(e^{2x}) = -\frac{1}{1+(e^{2x})^2}\cdot\frac{d}{dx}e^{2x}$$

$$= -\frac{2e^{2x}}{1+e^{4x}}$$

$$\frac{d}{dx}\left(\cos^{-1}(7x+5)\right)^2 = 2\cos^{-1}(7x+5)\,\frac{d}{dx}\cos^{-1}(7x+5)$$

$$= 2\cos^{-1}(7x+5)\cdot(-1)\frac{7}{\sqrt{1-(7x+5)^2}}$$

$$= \frac{-14\cos^{-1}(7x+5)}{\sqrt{1-(7x+5)^2}}$$

$$\frac{d}{dx}\sin^{-1}(\cos^{-1}(\sec^{-1}x))$$

$$= \frac{1}{\sqrt{1-(\cos^{-1}(\sec^{-1}x))^2}}\frac{d}{dx}\cos^{-1}(\sec^{-1}x)$$

$$= \frac{1}{\sqrt{1-(\cos^{-1}(\sec^{-1}x))^2}}\frac{-1}{\sqrt{1-(\sec^{-1}x)^2}}\cdot\frac{1}{x\sqrt{x^2-1}}$$

ทำแบบฝึกหัด 2.10

1. คี่ 2. คี่ 3.1

2.11 ฟังก์ชันไฮเพอร์โบลิก

นิยาม

hyperbolic sine $\sinh x = \dfrac{e^{x}-e^{-x}}{2}$

hyperbolic cosine $\cosh x = \dfrac{e^{x}+e^{-x}}{2}$

hyperbolic tangent $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

hyperbolic cotanget $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$

hyperbolic secant $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^{x}+e^{-x}}$

hyperbolic cosecant $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^{x}-e^{-x}}$

cosec x

เอกลักษณ์

$$\cosh^2 x - \sinh^2 x = 1$$

Pf:
$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$
$$= \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}}}{4} - \frac{\cancel{e^{2x}} - 2 + \cancel{e^{-2x}}}{4}$$
$$= 1 \qquad \square$$

2.12 อนุพันธ์ของฟังก์ชันไฮเพอร์โบลิก

Thm
$$\frac{d}{dx}\sinh x = \cosh x$$
$$\frac{d}{dx}\cosh x = \sinh x$$

Pf:
$$\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)$$
$$= \frac{e^x + e^{-x}}{2}$$
$$= \cosh x \qquad \square$$

ถ้า $u = g(x)$ เป็นฟังก์ชันที่หาอนุพันธ์ได้ โดยกฎลูกโซ่จะได้

$$\frac{d}{dx}\sinh u = \cosh u \frac{du}{dx} \qquad \frac{d}{dx}\cosh u = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}\operatorname{cosech} u = -\operatorname{cosech} u \coth u \frac{du}{dx}$$

$$\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}\tanh u = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}\coth u = -\operatorname{cosech}^2 u \frac{du}{dx}$$

Ex

$$\frac{d}{dx} \sinh(x^2+5x) = \cosh(x^2+5x)\cdot(2x+5)$$

$$\frac{d}{dx} \tanh^7(\sinh x) = 7\tanh^6(\sinh x)\frac{d}{dx}\tanh(\sinh x)$$

$$= 7\tanh(\sinh x)(-1)\,\mathrm{sech}^2(\sinh x)\cosh x$$