ชุดฝึกหัด 11

อนุพันธ์ของฟังก์ชันเชิงกำลังและฟังก์ชันลอการิทึม

1. **จงหาอนุพันธ์ของฟังก์ชันที่กำหนดในแต่ละข้อต่อไปนี้**

1.1 $y = x^{\pi}$

1.2 $y = \pi^{x}$

1.3 $y = 2^{-x} \sin x$

1.4 $y = 2^{x} \cdot 3^{x^2} \cdot 5^{x^3}$

1.5 $y = \sqrt{\ln x}$

1.6 $y = \ln(\sin x)$

1.7 $y = (x^2 - 1)^{\sqrt{3}}$

1.8 $y = x^2 \ln \sqrt{\sin 2x}$

1.9 $y = \pi^{\pi^3} \cdot x^{\pi^3} \cdot \pi^{x^3}$

1.10 $y = \ln(\ln x)$

1.11 $y = \ln[x + \sqrt{1 + x^2}]$

1.12 $y = x \ln(x^2 + 5x + 1)$

1.13 $y = \ln(\ln \frac{1}{x})$

1.14 $y = \frac{1}{2} \ln(4x) - 5e^{2x}$

1.15 $y = \frac{1}{2} \ln (\frac{1+x^2}{1-x^2})$

1.16 $y = \ln \left[\frac{(x^2+1)^2}{x\sqrt{x^2-1}} \right]$

1.17 $y = \ln \left[\frac{x\sqrt{3x-1}}{(x^2+1)^3} \right]$

1.18 $y = \ln (\sqrt{x+1} + \sqrt{x})$

1.19 $y = \ln [\sqrt{x+a} + \sqrt{x}]$

1.20 $y = \ln (x + \sqrt{x^2+a^2})$

1.21 $y = \ln(\sin e^{x})$

1.22 $y = \ln(x^2 \ln x)$

1.23 $y = \ln(e^{ax} + e^{-ax})$

1.24 $y = \ln(\tan^2 x)$

1.25 $y = \sin(e^{\sqrt{x^2+4}})$

1.26 $y = x^{\ln x}$

1.27 $y = x^{\ln(\ln x)}$

1.28 $y = x^{x^2}$

1.29 $y = (x^2+1)^{2x}$, $x > 0$

1.30 $y = (\sin x)^{x}$

1.31 $y = (x^2+1)^{ex}$

1.32 $y = 2^{\sin x}$

1.33 $y = (\sin x)^{\cos x}$, $\sin x > 0$

1.34 $y = (\sin x)^{\tan x}$, $\sin x > 0$

1.35 $y = \sin^{-1} (e^{x})$

1.36 $y = e^{\tan^{-1}(x^2/\sqrt{3})}$

1.37 $y = \arccos \left[\ln^2 (\frac{x^2+3x+1}{x^3-4x^2}) \right]$

1.38 $y = \ln(\tan^{-1} x)$

1.39 $y = x^{\cos x}$

1.40 $y = \sin (x\sqrt{x})$

1.41 $y = \ln \sqrt{\sin^{-1} x}$

1.42 $y = \sin^{-1} (\ln x)$

1.43 $y = \cos(2^{x})$

1.44 $y = x^{\ln(\sin x)}$

1.45 $y = 3^{\tan x}$

1.46 $y = 4^{\cot x}$

1.47 $y = \ln(\cot^2 x)$

1.48 $y = (\sqrt{3})^{\cos x}$

1.49 $y = e^{\csc^2 x}$

1.50 $y = x \tan^{-1}(x/a) - \frac{a}{x}\ln(x^2+a^2)$

1.51 $y = \frac{1}{a}\ln\left(\frac{x}{x+\sqrt{a^2-x^2}}\right)$

1.52 $y = \log_{10} e^x$

1.53 $y = \log_3(1+x^2)$

1.54 $y = \log_2(x^2-2x+1)$

1.55 $y = \log_2(1+x^2)$

1.56 $y = \frac{1}{a}\log_b\left(\frac{x}{\sqrt[3]{x^2+a^2}}\right)$

1.57 $y = (\sin x)^{x^{\cos x}}$

1.58 $y = (\tan x)^{x^{\sin x}}$

1.59 $y = (\sin x)^{\tan x^{\cos x}}$

1.60 $y = x^{\tan x^{\cos x}}$

2. จงใช้คุณสมบัติของฟังก์ชันลอการิทึม ในการหา $\frac{dy}{dx}$ ของข้อต่อไปนี้

2.1 $y = \frac{\sqrt{x}\,(x^3+2)^2}{\sqrt[3]{3x+4}}$

2.2 $y = \frac{x \cos x}{(x^2+1)^3 \sin x}$

2.3 $y = (x \sin x)(\cos x)(\ln x)$

2.4 $y = (x^3 + \cot^{-1} x)^{\sec x^2}$

3. จงหาอนุพันธ์ $\frac{dy}{dx}$ ของฟังก์ชันอิมพลิซิทต่อไปนี้

3.1 $x \ln y + y \ln x = 2$

3.2 $2^{xy} = x$

3.3 $\ln(x^2+y^2) = x+y$

3.4 $\ln(x^2-y^2) = x-y$

3.5 $x^y = 4$

3.6 $e^{x+y} = y$

3.7 $\ln(\frac{y}{x}) - \ln(\frac{x}{y}) = 1$

3.8 $\ln(\ln y) = e^x$

3.9 $e^x \sin y + e^y \sin x = 4$

3.10 $e^y \cos x + e^{-x} \sin y = 10$

3.11 $\ln x + \ln y = x \cos y$

3.12 $x = \ln(\csc y + \cot y)$

3.13 $2^y = (\sin^5 7x)^{(3x^2-1)}$

3.14 $xy = \ln(\sin(x+y))$

4. จงหาอนุพันธ์ $\frac{d}{dx}|\ln x|$

5. ถ้า $y = \tan u$, $u = v - \frac{1}{v}$ และ $v = \ln x$ จงหาค่าของ $\frac{dy}{dx}$ ที่ $x = e$

6. กำหนดให้ $f(x) = (x^2+1)^{(2-3x)}$ จงหา $f'(1)$

7. จงหาค่าของ $\lim_{h \to 0}\left[\frac{1}{h}\ln\left(\frac{2+h}{2}\right)\right]$

8. กำหนดให้ g เป็นฟังก์ชันผกผันของ f และ $f(x) = e^{x^3+x^2+x}$ จงหาค่าของ $g'(e^3)$

9. จงหาสมการเส้นสัมผัสเส้นโค้ง $\sin^2(xy) + \ln[\tan(xy^3)] = 0$ ที่จุด $(\frac{\pi}{4}, 1)$

10. จงหาจำนวนจริง m ที่จะทำให้เส้นตรง $y = mx$ สัมผัสกราฟ $y = \ln x$

11. ถ้า f และ g เป็นฟังก์ชันที่มีอนุพันธ์ และ $f(x) > 0$ ทุก x จงแสดงว่า

$$\frac{d}{dx} f(x)^{g(x)} = g(x)[f(x)]^{(g(x)-1)} f'(x) + f(x)^{g(x)}[\ln f(x)]\, g'(x)$$

เฉลยชุดฝึกหัด 11

1 1.1 $y' = \pi x^{\pi-1}$, π เป็นจำนวนจริง

1.2 $y' = (\pi^x)' = \pi^x \ln \pi$ (π เป็นค่าคงที่)

1.3 $y' = (2^{-x})' \sin x + 2^{-x}(\sin x)' = -2^{-x} \sin x \ln 2 + 2^{-x} \cos x$

1.4 $y' = (2^x)' . 3^{x^2} . 5^{x^3} + 2^x . (3^{x^2})' . 5^{x^3} + 2^x . 3^{x^2} . (5^{x^3})'$

$= 2^x . \ln 2 . 3^{x^2} . 5^{x^3} + 2^x 3^{x^2}(2x) . \ln 3 . 5^{x^3} + 2^x . 3^{x^2} . 5^{x^3}(3x^2) \ln 5$

$= 2^x . 3^{x^2} . 5^{x^3}(\ln 2 + 2x \ln 3 + 3x^2 \ln 5)$

1.5 $y' = \frac{d}{dx}\sqrt{\ln x} = \frac{d\sqrt{\ln x}}{d(\ln x)} . \frac{d \ln x}{dx} = \frac{1}{2\sqrt{\ln x}} . \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$

1.6 $y' = \frac{d}{dx} \ln (\sin x) = \frac{d \ln (\sin x)}{d(\sin x)} . \frac{d \sin x}{dx} = \frac{\cos x}{\sin x} = \cot x$

1.7 $y' = \frac{d}{dx}(x^2 - 1)^{\sqrt{3}} = \sqrt{3}(x^2 - 1)^{\sqrt{3}-1} . \frac{d}{dx}(x^2 - 1) = 2\sqrt{3}\, x(x^2 - 1)^{\sqrt{3}-1}$

1.8 $y' = (x^2)' \ln \sqrt{\sin 2x} + x^2(\ln \sqrt{\sin 2x})'$

$= 2x \ln \sqrt{\sin 2x} + x^2 . \frac{1}{\sqrt{\sin 2x}} . \frac{1}{2\sqrt{\sin 2x}} . \cos 2x . 2$

$= 2x \ln \sqrt{\sin 2x} + \frac{x^2 \cos 2x}{\sin 2x} = 2x \ln \sqrt{\sin 2x} + x^2 \cot 2x$

1.9 $y' = \left(\pi^{\pi^3}\right)' x^{\pi^3} . \pi^{x^3} + \pi^{\pi^3} . \left(x^{\pi^3}\right)' . \pi^{x^3} + \pi^{\pi^3} . x^{\pi^3} \left(\pi^{x^3}\right)'$

$= 0 + \pi^{\pi^3} (\pi^3) x^{\pi^3-1} . \pi^{x^3} + \pi^{\pi^3} . x^{\pi^3} . \pi^{x^3} (3x^2) . \ln\pi$

$= \pi^{\pi^3+3} . \pi^{x^3} x^{\pi^3-1} + 3\pi^{\pi^3} \ln\pi . x^2 . x^{\pi^3} . \pi^{x^3}$

1.10 $y' = \frac{d}{dx} \ln(\ln x) = \frac{d \ln (\ln x)}{d(\ln x)} . \frac{d \ln x}{dx} = \frac{1}{\ln x} . \frac{1}{x} = \frac{1}{x \ln x}$

1.11 $y' = \frac{d}{dx} \ln[x + \sqrt{1+x^2}] = \frac{1}{x + \sqrt{1+x^2}} . \frac{d}{dx}(x + \sqrt{1+x^2})$

$= \frac{1}{x + \sqrt{1+x^2}} . (1 + \frac{2x}{2\sqrt{1+x^2}}) = \frac{1}{x + \sqrt{1+x^2}} . (\frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}}) = \frac{1}{\sqrt{1+x^2}}$

1.12 $y' = \ln(x^2 + 5x + 1) + \frac{x}{x^2 + 5x + 1} . (2x+5) = \ln(x^2 + 5x + 1) + \frac{x(2x+5)}{x^2 + 5x + 1}$

1.13 $y' = \frac{d}{dx}\ln(\ln\frac{1}{x}) = \frac{1}{\ln(1/x)}\frac{d}{dx}\ln(\frac{1}{x})$

$= \frac{1}{-\ln x}\cdot(\frac{1}{1/x})\cdot\frac{d}{dx}(\frac{1}{x}) = \frac{-x}{\ln x}\cdot(\frac{-1}{x^2}) = \frac{1}{x\ln x}$

1.14 $y' = \frac{d}{dx}[\frac{1}{2}\ln(4x) - 5e^{2x}] = \frac{1}{2}\frac{d}{dx}\ln(4x) - 5\frac{d}{dx}e^{2x}$

$= \frac{1}{2}\cdot\frac{1}{4x}\frac{d(4x)}{dx} - 5e^{2x}\frac{d(2x)}{dx} = \frac{1}{2x} - 10e^{2x}$

1.15 $y' = \frac{d}{dx}\left[\frac{1}{2}\ln(\frac{1+x^2}{1-x^2})\right] = \frac{1}{2}\frac{d}{dx}\left[\ln(1+x^2) - \ln(1-x^2)\right]$

$= \frac{1}{2}\left[\frac{d}{dx}\ln(1+x^2) - \frac{d}{dx}\ln(1-x^2)\right] = \frac{1}{2}\left[\frac{1}{1+x^2}\frac{d}{dx}(1+x^2) - \frac{1}{1-x^2}\cdot\frac{d}{dx}(1-x^2)\right]$

$= \frac{1}{2}\left[\frac{2x}{1+x^2} - \frac{(-2x)}{1-x^2}\right] = x\left[\frac{1}{1+x^2} + \frac{1}{1-x^2}\right] = \frac{2x}{1-x^4}$

1.16 $y' = \frac{d}{dx}\ln\frac{(x^2+1)^2}{x\sqrt{x^2-1}} = \frac{d}{dx}\left[2\ln(x^2+1) - \ln x - \frac{1}{2}\ln(x^2-1)\right]$

$= \frac{2}{x^2+1}\cdot(x^2+1)' - \frac{1}{x} - \frac{1}{2(x^2-1)}\cdot(x^2-1)' = \frac{2(2x)}{x^2+1} - \frac{1}{x} - \frac{2x}{2(x^2-1)}$

$= \frac{4x^2(x^2-1) - (x^4-1) - x^2(x^2+1)}{x(x^4-1)} = \frac{2x^4-5x^2+1}{x(x^4-1)}$

1.17 $y' = \frac{d}{dx}\left[\ln x + \frac{1}{2}\ln(3x-1) - 3\ln(x^2+1)\right]$

$= \frac{1}{x} + \frac{1}{2(3x-1)}\cdot(3x-1)' - \frac{3}{x^2+1}(x^2+1)' = \frac{1}{x} + \frac{3}{2(3x-1)} - \frac{6x}{x^2+1}$

1.18 $y' = \frac{d}{dx}\ln(\sqrt{x+1} + \sqrt{x}) = \frac{1}{\sqrt{x+1}+\sqrt{x}}\cdot(\sqrt{x+1}+\sqrt{x})'$

$= \frac{1}{\sqrt{x+1}+\sqrt{x}}(\frac{1}{2\sqrt{x+1}} + \frac{1}{2\sqrt{x}}) = \frac{1}{\sqrt{x+1}+\sqrt{x}}(\frac{\sqrt{x}+\sqrt{x+1}}{2\sqrt{x(x+1)}}) = \frac{1}{2\sqrt{x(x+1)}}$

1.19 $y' = \frac{d}{dx}\ln(\sqrt{x+a} + \sqrt{x}) = \frac{1}{\sqrt{x+a}+\sqrt{x}}(\sqrt{x+a}+\sqrt{x})'$

$= \frac{1}{\sqrt{x+a}+\sqrt{x}}(\frac{1}{2\sqrt{x+a}} + \frac{1}{2\sqrt{x}}) = \frac{1}{\sqrt{x+a}+\sqrt{x}}(\frac{\sqrt{x}+\sqrt{x+a}}{2\sqrt{x(x+a)}}) = \frac{1}{2\sqrt{x(x+a)}}$

1.20 $y' = \frac{d}{dx}\ln(x + \sqrt{x^2+a^2}) = \frac{1}{x+\sqrt{x^2+a^2}}\cdot(x+\sqrt{x^2+a^2})'$

$= \frac{1}{x+\sqrt{x^2+a^2}}(1 + \frac{2x}{2\sqrt{x^2+a^2}}) = \frac{1}{x+\sqrt{x^2+a^2}}(\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}) = \frac{1}{\sqrt{x^2+a^2}}$

1.21 $y' = \frac{d}{dx}\ln(\sin e^x) = \frac{1}{\sin e^x}\cdot\frac{d}{dx}\sin e^x = \frac{1}{\sin e^x}\cdot\cos e^x\,\frac{d}{dx}e^x$

$= \frac{e^x \cos e^x}{\sin e^x} = e^x \cot e^x$

1.22 $y' = \frac{d}{dx}[\ln x^2 + \ln(\ln x)] = \frac{d}{dx}[2\ln x + \ln(\ln x)] = \frac{2}{x} + \frac{1}{x\ln x}$

1.23 $y' = \frac{d}{dx}\ln(e^{ax} + e^{-ax}) = \frac{1}{e^{ax} + e^{-ax}}(e^{ax} + e^{-ax})'$

$= \frac{1}{e^{ax} + e^{-ax}}(e^{ax}\,.\,a + e^{-ax}(-a)) = a\,.\,\frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$

1.24 $y' = \frac{d}{dx}\ln(\tan^2 x) = \frac{1}{\tan^2 x}\frac{d}{dx}\tan^2 x$

$= \frac{1}{\tan^2 x}\,.\,2\tan x\,\frac{d}{dx}\tan x = \frac{2\tan x\sec^2 x}{\tan^2 x} = \frac{4}{\sin 2x}$

1.25 $y' = \frac{d}{dx}\sin(e^{\sqrt{x^2+4}}) = \cos(e^{\sqrt{x^2+4}})\frac{d}{dx}e^{\sqrt{x^2+4}}$

$= \cos(e^{\sqrt{x^2+4}})\,e^{\sqrt{x^2+4}}\,.\,\frac{d}{dx}\sqrt{x^2+4} = \cos(e^{\sqrt{x^2+4}})\,e^{\sqrt{x^2+4}}\,.\,\frac{x}{\sqrt{x^2+4}}$

1.26 จาก $y = x^{\ln x}$ จะได้ $\ln y = (\ln x)^2$ และได้

$\frac{y'}{y} = \frac{2\ln x}{x}$ หรือ $y' = x^{\ln x}\,(\frac{\ln x}{x})$

1.27 จาก $y = x^{\ln(\ln x)}$ ทำให้ได้ $\ln y = \ln(\ln x)\,.\,\ln x$

$\therefore\ \frac{1}{y}\,.\,y' = (\ln x)'\ln(\ln x) + \ln x\,(\ln(\ln x))' = \frac{\ln(\ln x)}{x} + \frac{\ln x}{\ln x}\,.\,\frac{1}{x}$

หรือ $y' = x^{\ln(\ln x)}\,(\frac{1}{x} + \frac{\ln(\ln x)}{x})$

1.28 $y = x^{x^2} \Rightarrow \ln y = x^2\ln x$

$\frac{1}{y}\,.\,y' = (x^2)'\ln x + x^2(\ln x)' = 2x\ln x + x$

$y' = x^{x^2}(2x\ln x + x)$

1.29 $y = (x^2+1)^{2x}$, $x > 0$ $\Rightarrow \ln y = 2x\ln(x^2+1)$

$\frac{1}{y}\,.\,y' = (2x)'\ln(x^2+1) + 2x(\ln(x^2+1))'$

$\therefore\ y' = (x^2+1)^{2x}\left[2\ln(x^2+1) + \frac{4x^2}{x^2+1}\right]$

1.30 $y = (\sin x)^x \Rightarrow \ln y = x \ln(\sin x)$

$\frac{1}{y} \cdot y' = (x)' \ln(\sin x) + x (\ln(\sin x))' = \ln(\sin x) + \frac{x \cos x}{\sin x}$

$\therefore \quad y' = (\sin x)^x (\ln (\sin x) + x \cot x)$

1.31 $y = (x^2 + 1)^{e^x} \Rightarrow \ln y = e^x \ln(x^2 + 1)$

$\frac{1}{y} y' = e^x \ln(x^2 + 1) + e^x \cdot \frac{2x}{x^2 + 1}$

$\therefore y' = (x^2 + 1)^{e^x} [e^x \ln(x^2 + 1) + \frac{2xe^x}{x^2 + 1}]$

1.32 $y' = 2^{\sin x} \cdot \ln 2 \frac{d}{dx} \sin x = 2^{\sin x} \cdot \cos x \ln 2$

1.33 $y = (\sin x)^{\cos x}, \quad \sin x > 0 \Rightarrow \ln y = \cos x \ln(\sin x)$

$\frac{1}{y} \cdot y' = (\cos x)' \ln(\sin x) + \cos x (\ln(\sin x))' = - \sin x \ln(\sin x) + \frac{\cos^2 x}{\sin x}$

$\therefore \quad y' = (\sin x)^{\cos x} [\cos x \cot x - \sin x \ln(\sin x)]$

1.34 $y = (\sin x)^{\tan x}, \sin x > 0 \Rightarrow \ln y = \tan x \ln(\sin x)$

$\frac{1}{y} \cdot y' = (\tan x)' \ln(\sin x) + \tan x(\ln(\sin x))'$

$= \sec^2 x \ln(\sin x) + \frac{\tan x}{\sin x} \cdot \cos x = 1 + \sec^2 x \ln(\sin x)$

$\therefore \quad y' = (\sin x)^{\tan x} [1 + \sec^2 x \ln(\sin x)]$

1.35 $y' = \frac{d}{dx} \sin^{-1} (e^x) = \frac{1}{\sqrt{1 - e^{2x}}} \frac{de^x}{dx} = \frac{e^x}{\sqrt{1 - e^{2x}}}$

1.36 $y' = \left(e^{\tan^{-1}(x^2/\sqrt{3})}\right)' = e^{\tan^{-1}(x^2/\sqrt{3})} \frac{d}{dx} (\tan^{-1} \frac{x^2}{\sqrt{3}})$

$= e^{\tan^{-1}(x^2/\sqrt{3})} \cdot \frac{1}{1 + x^4/3} \cdot \frac{d}{dx} (\frac{x^2}{\sqrt{3}})$

$= e^{\tan^{-1}(x^2/\sqrt{3})} \cdot \frac{3}{3+ x^4} \cdot \frac{2x}{\sqrt{3}} = \frac{2\sqrt{3}\, x}{3+x^4} \cdot e^{\tan^{-1}(x^2/\sqrt{3})}$

1.37 $y' = -\dfrac{1}{\sqrt{1 - \ln^4 (\frac{x^2+3x+1}{x^3-4x^2})}} \cdot 2 \ln (\frac{x^2+3x+1}{x^3-4x^2})\left[\dfrac{2x+3}{x^2+3x+1} - \dfrac{3x^2-8x}{x^3-4x^2}\right]$

1.38 $y' = \dfrac{1}{\tan^{-1} x} \cdot \dfrac{1}{1+x^2} = \dfrac{1}{(1+x^2)\tan^{-1} x}$

1.39 $y = x^{\cos x} \Rightarrow \ln y = \cos x . \ln x$

$\therefore \dfrac{1}{y} . y' = (\cos x)' \ln x + \cos x(\ln x)' = (- \sin x) \ln x + \dfrac{\cos x}{x}$

$\therefore y' = x^{\cos x} (\dfrac{\cos x}{x} - \sin x . \ln x)$

1.40 ให้ $u = x^{\sqrt{x}}$ แล้ว $\ln u = \sqrt{x} \ln x$ ทำให้ได้

$\dfrac{1}{u} . u' = \dfrac{1}{2\sqrt{x}} . \ln x + \dfrac{\sqrt{x}}{x}$ และได้ $u' = \dfrac{x^{\sqrt{x}}}{\sqrt{x}} (\ln\sqrt{x} + 1)$

$\therefore y' = \cos (x^{\sqrt{x}}) \dfrac{du}{dx} = \cos (x^{\sqrt{x}}) . \dfrac{x^{\sqrt{x}}}{\sqrt{x}} (\ln\sqrt{x} + 1)$

1.41 $y = \ln\sqrt{\sin^{-1} x} = \dfrac{1}{2} \ln(\sin^{-1} x)$

$y' = \dfrac{1}{2 \sin^{-1} x} \dfrac{d}{dx} (\sin^{-1} x) = \dfrac{1}{2\sqrt{1-x^2} . \sin^{-1} x}$

1.42 $y' = \dfrac{1}{\sqrt{1 - \ln^2 x}} \cdot \dfrac{1}{x} = \dfrac{1}{x\sqrt{1 - \ln^2 x}}$

1.43 $y' = - \sin (2^x) \dfrac{d\, 2^x}{dx} = - 2^x \ln 2 . \sin (2^x)$

1.44 $y = x^{\ln(\sin x)} \Rightarrow \ln y = \ln (\sin x) . \ln x$

$\dfrac{1}{y} . y' = (\ln(\sin x))' \ln x + \ln (\sin x)(\ln x)'$

$= \dfrac{\cos x}{\sin x} . \ln x + \dfrac{\ln (\sin x)}{x} = \cot x . \ln x + \dfrac{\ln (\sin x)}{x}$

1.45 $y' = 3^{\tan x} . \ln 3 . \dfrac{d}{dx} \tan x = \sec^2 x . 3^{\tan x} . \ln 3$

1.46 $y' = 4^{\cot x} \ln 4 . (\cot x)' = - \csc^2 x . \ln 4 . 4^{\cot x}$

1.47 $y' = \dfrac{1}{\cot^2 x} (\cot^2 x)' = \dfrac{(2 \cot x)(- \csc^2 x)}{\cot^2 x} = \dfrac{-4}{\sin 2x} = - 4 \csc 2x$

1.48 $y' = (\sqrt{3})^{\cos x}(-\sin x)\,.\,\ln\sqrt{3} = -\sin x\,.\,\ln\sqrt{3}\,.\,(\sqrt{3})^{\cos x}$

1.49 $y' = e^{\csc^2 x}\,.\,(\csc^2 x)' = 2\csc x(-\csc x\cot x)\,.\,e^{\csc^2 x}$

$= -2\csc^2 x\,.\,\cot x\,.\,e^{\csc^2 x}$

1.50 $y' = (x)'\tan^{-1}(\frac{x}{a}) + x(\tan^{-1}\frac{x}{a})' - (\frac{a}{x})'\ln(x^2+a^2) - (\frac{a}{x})(\ln(x^2+a^2))'$

$= \tan^{-1}\frac{x}{a} + \frac{x}{a}\,.\,\frac{1}{1+x^2/a^2} + \frac{a}{x^2}\ln(x^2+a^2) - \frac{a\,.\,2x}{x(x^2+a^2)}$

$= \tan^{-1}\frac{x}{a} + \frac{xa}{x^2+a^2} + \frac{a}{x^2}\ln(x^2+a^2) - \frac{2a}{(x^2+a^2)}$

1.51 $y = \frac{1}{a}[\ln x - \ln(x+\sqrt{a^2-x^2})]$

$y' = \frac{1}{a}\left[\frac{1}{x} - \frac{1}{x+\sqrt{a^2-x^2}}(1+\frac{(-2x)}{2\sqrt{a^2-x^2}})\right] = \frac{1}{a}\left[\frac{1}{x} - \frac{\sqrt{a^2-x^2}-x}{\sqrt{a^2-x^2}\,(x+\sqrt{a^2-x^2})}\right]$

$= \frac{1}{a}\left[\frac{\sqrt{a^2-x^2}\,(x+\sqrt{a^2-x^2}) - x\sqrt{a^2-x^2} + x^2}{x\sqrt{a^2-x^2}\,(x+\sqrt{a^2-x^2})}\right] = \frac{a}{x\sqrt{a^2-x^2}\,(x+\sqrt{a^2-x^2})}$

1.52 $y' = (\log_{10} e^x)' = (\frac{\ln e^x}{\ln 10})' = (\frac{x}{\ln 10})' = \frac{1}{\ln 10} = \log e$

<u>หรือ</u> $y' = (\log e^x)' = \frac{1}{e^x}\,.\,\log e\,.\,(e^x)' = \log e$

1.53 $y' = \frac{d}{dx}\log_3(1+x^2) = \frac{1}{1+x^2}\,.\,\log_3 e\,.\,(1+x^2)' = \frac{2x\log_3 e}{1+x^2}$

1.54 $y' = [\log_2(x^2-2x+1)]' = \frac{(2x-2)\log_2 e}{x^2-2x+1} = \frac{2\log_2 e}{x-1}$

1.55 $y' = \frac{1}{1+x^2}\,.\,\log_2 e\,.2x = \frac{2x}{1+x^2}\,.\,\log_2 e$

1.56 $y = \frac{1}{a}\log_b\left(\frac{x}{\sqrt[3]{x^2+a^2}}\right) = \frac{1}{a}\left[\log_b x - \frac{1}{3}\log_b(x^2+a^2)\right]$

$\therefore\ y' = \frac{1}{a}\left[\frac{1}{x}\log_b e - \frac{1}{3(x^2+a^2)}\,.\,\log_b e\,.\,(x^2+a^2)'\right] = \frac{\log_b e}{a}\left[\frac{1}{x} - \frac{2x}{3(x^2+a^2)}\right]$

1.57 $y = (\sin x)^{x^{\cos x}} \Rightarrow \ln y = x^{\cos x} \ln(\sin x)$

$\therefore \frac{1}{y} y' = (x^{\cos x})' \ln(\sin x) + x^{\cos x} (\ln(\sin x))'$

แต่ $(\ln(\sin x))' = \cot x$ และ $(x^{\cos x})' = x^{\cos x} \left(\frac{\cos x}{x} - \sin x \,.\, \ln x\right)$

$\therefore y' = (\sin x)^{x^{\cos x}} \left[x^{\cos x} \ln(\sin x)\left(\frac{\cos x}{x} - \sin x \,.\, \ln x\right) + x^{\cos x} \,.\, \cot x\right]$

$= x^{\cos x} \,.\, (\sin x)^{x^{\cos x}} \left[\cot x + \frac{\cos x \,.\, \ln(\sin x)}{x} - \sin x \,.\, \ln x \,.\, \ln(\sin x)\right]$

1.58 $y = (\tan x)^{x^{\sin x}} \Rightarrow \ln y = x^{\sin x} \ln(\tan x)$

$\therefore \frac{1}{y} \,.\, y' = (x^{\sin x})' \ln(\tan x) + x^{\sin x} (\ln(\tan x))'$

แต่ถ้าให้ $u = x^{\sin x}$ เราจะได้ $\ln u = \sin x(\ln x)$ และดังนั้น

$u' = (x^{\sin x})' = x^{\sin x} \left(\frac{\sin x}{x} + \cos x \,.\, \ln x\right)$ และ

$(\ln(\tan x))' = \frac{1}{\tan x} \,.\, \sec^2 x = \frac{\sec^2 x}{\tan x}$ เพราะฉะนั้น

$y' = (\tan x)^{x^{\sin x}} \left[x^{\sin x} \left(\frac{\sin x}{x} + \cos x \,.\, \ln x\right) \ln(\tan x) + x^{\sin x} \,.\, \frac{\sec^2 x}{\tan x}\right]$

1.59 $y = (\sin x)^{\tan x^{\cos x}} \Rightarrow \ln y = \tan x^{\cos x} \ln(\sin x)$

$\frac{1}{y} \,.\, y' = (\tan x^{\cos x})' \ln(\sin x) + \tan x^{\cos x} [\ln(\sin x)]'$

แต่ถ้าให้ $u = \tan x^{\cos x}$ แล้ว $\ln u = \cos x \ln(\tan x)$ และดังนั้น

$u' = \tan x^{\cos x} \left[- \sin x \,.\, \ln(\tan x) + \frac{\cos x}{\tan x} \,.\, \sec^2 x\right]$

$= \tan x^{\cos x} [\csc x - \sin x \,.\, \ln(\tan x)]$

และ $[\ln(\sin x)]' = \frac{1}{\sin x} \,.\, \cos x = \cot x$ เพราะฉะนั้น

$y' = y \,.\, [\tan x^{\cos x} (\csc x - \sin x \ln(\tan x)) \ln(\sin x) + \tan x^{\cos x} \,.\, \cot x]$

$= \tan x^{\cos x} (\sin x)^{\tan x^{\cos x}} (\cot x + \csc x - \sin x) \ln(\tan x) \ln(\sin x))$

1.60 $y = x^{\tan x^{\sin x}} \Rightarrow \ln y = \tan x^{\sin x} \,.\, \ln x$

$\frac{1}{y} \,.\, y' = (\tan x^{\sin x})' \ln x + \frac{\tan x^{\sin x}}{x}$

แต่ถ้า $u = (\tan x)^{\sin x}$ แล้ว $\ln u = \sin x \ln(\tan x)$ และดังนั้น

$u' = \tan x^{\sin x} \left[\cos x \ln(\tan x) + \frac{\sin x}{\tan x} \,.\, \sec^2 x\right]$

$= \tan x^{\sin x} [\cos x \ln(\tan x) + \sec x]$

เพราะฉะนั้น

$$y' = x^{\tan x^{\sin x}} [\tan x^{\sin x} (\cos x \ln(\tan x) + \sec x) + \frac{\tan x^{\sin x}}{x}]$$

$$= \tan x^{\sin x} . x^{\tan x^{\sin x}} [\frac{1}{x} + \sec x + \cos x \ln(\tan x)]$$

2.

2.1 ถ้า $y = \frac{\sqrt{x}(x^3+2)^2}{\sqrt[3]{3x+4}}$ แล้ว $\ln y = \frac{1}{2}\ln x + 2\ln(x^3+2) - \frac{1}{3}\ln(3x+4)$ ซึ่งจะทำให้ได้

$$\frac{1}{y}.y' = \frac{1}{2x} + \frac{2}{x^3+2}.3x^2 - \frac{3}{3(3x+4)} = \frac{1}{2x} + \frac{6x^2}{x^3+2} - \frac{1}{3x+4}$$

เพราะฉะนั้น $y' = \frac{\sqrt{x}(x^3+2)^2}{\sqrt[3]{3x+4}}(\frac{1}{2x} + \frac{6x^2}{x^3+2} - \frac{1}{3x+4})$

2.2 ถ้า $y = \frac{x \cos x}{(x^2+1)^3 \sin x}$ แล้ว $\ln y = \ln x + \ln(\cos x) - 3\ln(x^2+1) - \ln(\sin x)$

ซึ่งจะทำให้ได้ $\frac{1}{y}.y' = \frac{1}{x} - \frac{\sin x}{\cos x} - \frac{3(2x)}{x^2+1} - \frac{\cos x}{\sin x} = \frac{1}{x} - \tan x - \frac{6x}{x^2+1} - \cot x$

เพราะฉะนั้น $y' = \frac{x \cos x}{(x^2+1)^3 \sin x}(\frac{1}{x} - \frac{6x}{x^2+1} - \tan x - \cot x)$

2.3 ถ้า $y = (x \sin x)(\cos x)(\ln x)$ แล้ว $\ln y = \ln x + \ln(\sin x) + \ln(\cos x) + \ln(\ln x)$

ซึ่งจะทำให้ได้ $\frac{1}{y}.y' = \frac{1}{x} + \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} + \frac{1}{x \ln x}$

เพราะฉะนั้น $y' = (x \sin x)(\cos x)(\ln x)(\frac{1}{x} + \frac{1}{x \ln x} + \cot x - \tan x)$

2.4 ถ้า $y = (x^3 + \cot^{-1} x)^{\sec x^2}$ แล้ว $\ln y = \sec x^2 \ln(x^3+\cot^{-1}x)$ แล้วจะได้

$$\frac{1}{y}.y' = (\sec x^2)' \ln(x^3 + \cot^{-1} x) + \sec x^2 [\ln(x^3 + \cot^{-1} x)]'$$

$$= 2x \sec x^2 \tan x^2 \ln(x^3 + \cot^{-1} x) + \frac{\sec x^2}{x^3 + \cot^{-1} x}[3x^2 + \frac{-1}{1+x^2}]$$

เพราะฉะนั้น

$$y' = (x^3 + \cot^{-1} x)^{\sec x^2}\left[2x \sec x^2 \tan x^2 \ln(x^3 + \cot^{-1} x) + \frac{(3x^4 + 3x^2 - 1)\sec x^2}{(1+x^2)(x^3 + \cot^{-1} x)}\right]$$

3.

3.1 $\ln y + \frac{x}{y} \cdot y' + y' \ln x + \frac{y}{x} = 0 \Rightarrow (\frac{x}{y} + \ln x)\, y' = -(\frac{y}{x} + \ln y)$

$\Rightarrow y' = (-\frac{y}{x})(\frac{y + x\ln y}{x + y\ln x}) = -\frac{y^2}{x^2} \cdot \frac{\ln y}{\ln x}$

3.2 $2^{xy} \cdot \ln 2(y + xy') = 1 \Rightarrow y' = \frac{1 - y\,2^{xy}\ln 2}{x\,2^{xy}\ln 2}$

3.3 $\frac{2x + 2yy'}{x^2 + y^2} = 1 + y' \Rightarrow 2x - x^2 - y^2 = (x^2 + y^2 - 2y)y' \quad \therefore\; y' = \frac{2x - x^2 - y^2}{x^2 + y^2 - 2y}$

3.4 $\frac{2x - 2yy'}{x^2 - y^2} = 1 - y' \Rightarrow x^2 - y^2 - 2x = (x^2 - y^2 - 2y)y' \quad \therefore\; y' = \frac{x^2 - y^2 - 2x}{x^2 - y^2 - 2y}$

3.5 $x^y = 4 \Rightarrow y\ln x = \ln 4 \quad \therefore\; y'\ln x + \frac{y}{x} = 0$ หรือ $y' = -\frac{y}{x\ln x}$

3.6 $e^{x+y}(1+y') = y' \Rightarrow (e^{x+y} - 1)y' = -e^{x+y} \quad \therefore\; y' = \frac{e^{x+y}}{1 - e^{x+y}}$

3.7 $\ln(\frac{y}{x}) - \ln(\frac{x}{y}) = 1 \Rightarrow \ln y - \ln x - \ln x + \ln y = 1 \Rightarrow 2\ln y - 2\ln x = 1$

ดังนั้น $\frac{2}{y} \cdot y' - \frac{2}{x} = 0$ เพราะฉะนั้น $y' = (\frac{2}{x})(\frac{y}{2}) = \frac{y}{x}$

3.8 หาอนุพันธ์เทียบกับ x ทั้ง 2 ข้างของสมการ $\ln(\ln y) = e^x$ จะได้ $\frac{y'}{y\ln y} = e^x$

$\therefore\; y' = y\ln y \cdot e^x$

3.9 $e^x \sin y + e^x \cos y\, y' + e^y \sin x \cdot y' + e^y \cos x = 0$

หรือ $(e^x \cos y + e^y \sin x)y' = -(e^y \cos x + e^x \sin y)$

$\therefore\; y' = \frac{-(e^y \cos x + e^x \sin y)}{e^y \sin x + e^x \cos y}$

3.10 $e^y \cos x \cdot y' - e^y \sin x - e^{-x} \sin y + e^{-x} \cos y \cdot y' = 0$

หรือ $(e^y \cos x + e^{-x} \cos y)y' = e^y \sin x + e^{-x} \sin y$

$\therefore\; y' = \frac{e^y \sin x + e^{-x} \sin y}{e^y \cos x + e^{-x} \cos y}$

3.11 $\frac{1}{x}+\frac{1}{y}.y' = \cos y - x\sin y . y'$ หรือ $(\frac{1}{y} + x\sin y)y' = \cos y - \frac{1}{x}$

$\therefore\ y' = \frac{y}{x}(\frac{x\cos y - 1}{1+xy\sin y})$

3.12 จาก $1 = (\frac{-\csc y\cot y - \csc^2 y}{\csc y + \cot y})\,y'$ ทำให้ได้

$y' = -\frac{\csc y + \cot y}{\csc y(\cot y + \csc y)} = -\frac{1}{\csc y} = -\sin y$

3.13 $2^y = (\sin^5 7x)^{(3x^2-1)} \Rightarrow y\ln 2 = (3x^2-1)\ln(\sin^5 7x)$

เมื่อหาอนุพันธ์เทียบกับ x ทั้งสองข้างของสมการจะได้

$y'\ln 2 = 6x\ln(\sin^5 7x) + \frac{3x^2-1}{\sin^5 7x}.5\sin^4 7x.\cos 7x.7$

$= 6x\ln(\sin^5 7x) + \frac{7(3x^2-1)\cos 7x}{\sin 7x}$

$\therefore\ y' = \frac{1}{\ln 2}[6x\ln(\sin^5 7x) + 7(3x^2-1)\cot 7x]$

3.14 $y + xy' = \frac{\cos(x+y)}{\sin(x+y)}(1+y') = \cot(x+y)(1+y')$

หรือ $(x - \cot(x+y))y' = \cot(x+y) - y$

$\therefore\ y' = \frac{\cot(x+y) - y}{x - \cot(x+y)}$

4. $\frac{d}{dx}|\ln x| = \frac{\ln x}{|\ln x|}.(\ln x)' = \frac{\ln x}{x|\ln x|}$

5. $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx} = \frac{d}{du}(\tan u).\frac{d}{dv}(v-\frac{1}{v}).\frac{d}{dx}(\ln x) = \sec^2 u.(1+\frac{1}{v^2}).\frac{1}{x}$

$x = e \Rightarrow v = \ln e = 1 \Rightarrow u = v - \frac{1}{v} = 0$

เพราะฉะนั้น $\frac{dy}{dx}\Big|_{x=e} = \sec^2(0)(1-\frac{1}{1^2}).\frac{1}{e} = (1)(0)(\frac{1}{e}) = 0$

6. ถ้า $f(x) = (x^2+1)^{(2-3x)}$ จะได้ $\ln f(x) = (2-3x)\ln(x^2+1)$

เมื่อหาอนุพันธ์เทียบกับ x ทั้งสองข้างของสมการจะได้ $\frac{f'(x)}{f(x)} = -3\ln(x^2+1) + \frac{(2-3x)(2x)}{x^2+1}$

เพราะฉะนั้น $f'(x) = (x^2+1)^{(2-3x)}\left[-3\ln(x^2+1) + \frac{2x(2-3x)}{x^2+1}\right]$ ซึ่งจะทำให้ได้

$f'(1) = (1+1)^{(2-3)}\left[-3\ln(1+1) + \frac{(2)(1)(2-3)}{1+1}\right] = 2^{-1}(\ln 2^{-3} - \frac{2}{2}) = -(\frac{1+3\ln 2}{2})$

7. $\lim_{h\to 0}\left[\frac{1}{h}\ln(\frac{2+h}{2})\right] = \lim_{h\to 0}\frac{1}{h}[\ln(2+h) - \ln 2] = \lim_{h\to 0}\frac{\ln(2+h) - \ln 2}{h}$

$= (\frac{d}{dx}\ln x)\Big|_{x=2} = (\frac{1}{x})\Big|_{x=2} = \frac{1}{2}$

8. g เป็นฟังก์ชันผกผันของ f ดังนั้น $y = g(x)$ ก็ต่อเมื่อ $x = f(y)$ ซึ่งจะทำให้ได้
$1 = f'(y)\ y' = f'(y)\ g'(x)$ หรือ $g'(x) = \frac{1}{f'(y)}$ ดังนั้นเมื่อต้องการหาค่า $g'(e^3)$
เราอาจหาค่า $f'(y) = f'(g(x))$ โดยการหาค่า y เมื่อ $x = e^3$ เสียก่อน โดยพิจารณาจาก

$$x = e^3 \iff e^3 = f(y) = e^{y^3+y^2+y} \iff 3 = y^3+y^2+y$$

$$\iff y^3+y^2+y-3 = 0 \iff (y-1)(y^2+2y+3) = 0$$

ดังนั้น $x = e^3$ ทำให้ได้ค่า y เป็นจำนวนจริง เมื่อ $y = 1$ เราจึงได้ $g'(e^3) = \frac{1}{f'(1)}$

แต่ $f'(x) = \frac{d}{dx} e^{x^3+x^2+x} = e^{x^3+x^2+x} . (3x^2+2x+1)$

ทำให้ได้ $f'(1) = e^3(3+2+1) = 6e^3$ เพราะฉะนั้น $g'(e^3) = \frac{1}{f'(1)} = \frac{1}{6e^3}$

9. เมื่อหาอนุพันธ์เทียบกับ x ทั้งสองข้างของสมการ $\sin^2 (xy) + \ln[\tan(xy^3)] = 0$ จะได้

$$2 \sin(xy) \cos(xy)[y + xy'] + \frac{\sec^2(xy^3)}{\tan(xy^3)}(y^3 + 3xy^2y') = 0 \quad (*)$$

เมื่อต้องการหา y' ที่จุด $(\frac{\pi}{4}, 1)$ เราจะแทนค่า $x = \frac{\pi}{4}$ และ $y = 1$ ลงใน (*) ทำให้ได้

$$2 \sin\frac{\pi}{4} \cos\frac{\pi}{4}(1 + \frac{\pi}{4}y') + \frac{\sec^2 (\pi/4)}{\tan(\pi/4)}(1 + \frac{3\pi}{4}y') = 0$$

หรือ $1 + \frac{\pi}{4}y' + 2(1 + \frac{3\pi}{4}y') = 0$ หรือ $\frac{7\pi}{4}y' = -3$ หรือ $y' = \frac{-12}{7\pi}$
เพราะฉะนั้นสมการเส้นสัมผัสที่ต้องการคือ

$$(y - 1) = \frac{-12}{7\pi}(x - \frac{\pi}{4}) \qquad \text{หรือ} \qquad 7\pi y + 12x - 10\pi = 0$$

10. สมมติให้เส้นตรง $y = mx$ สัมผัสกราฟ $y = \ln x$ ที่จุด (a, b) เราจะได้ว่า
(1) เนื่องจาก (a, b) อยู่บนเส้นตรง $y = mx$ ดังนั้น $b = ma$ หรือ $m = b/a$
(2) เนื่องจากความชัน m ของเส้นตรง $y = mx$ ซึ่งสัมผัส $y = \ln x$ ที่จุด (a, b) จะมีค่าเท่ากับ
$(\frac{d}{dx}\ln x)_{x=a}$ ดังนั้น $(\frac{1}{x})|_{x=a} = \frac{1}{a} = m = \frac{b}{a}$ ทำให้ได้ $b = 1$
(3) เนื่องจากจุด (a, b) อยู่บนกราฟ $y = \ln x$ ทำให้ได้ $b = \ln a$ หรือ $1 = \ln a$
นั่นคือได้ $a = e$

เพราะฉะนั้น $y = mx$ สัมผัส $y = \ln x$ ที่จุด (e, 1) ซึ่งทำให้ได้ความชัน $m = \frac{b}{a} = \frac{1}{e}$

11. ให้ $y = f(x)^{g(x)}$ แล้ว $\ln y = g(x) \ln[f(x)]$ ซึ่งเมื่อหาอนุพันธ์เทียบกับ x ตลอดสมการนี้จะได้

$\frac{1}{y} . y' = g'(x) \ln[f(x)] + \frac{g(x)}{f(x)} . f'(x)$ และได้

$$y' = f(x)^{g(x)}\left[g'(x) \ln(f(x)) + \frac{g(x)}{f(x)} . f'(x)\right]$$

$$= f(x)^{g(x)} . g'(x) \ln(f(x)) + f(x)^{g(x)-1} . g(x) . f'(x)$$