ชุดฝึกหัด 10

อนุพันธ์ของฟังก์ชันตรีโกณมิติ และตรีโกณมิติผกผัน

1. จงหาอนุพันธ์ของฟังก์ชันตรีโกณมิติที่กำหนดในข้อต่อไปนี้

1.1 $y = x \sec x + 2 \cot x$

1.2 $y = \dfrac{\sin x + \cos x}{\sin x - \cos x}$

1.3 $y = \dfrac{\csc x}{1 + x \cos x}$

1.4 $y = \dfrac{\csc x - \cot x}{\csc x + \cot x}$

1.5 $y = \tan^2 x$

1.6 $y = \sec^3 x$

1.7 $g(t) = \sin^2 t - \cos^2 t$

1.8 $f(t) = (\sin t + \cos t)^2$

1.9 $y = \sin 4x$

1.10 $y = \csc(x^3 + 1)$

1.11 $y = 2 \cos^2(x^2)$

1.12 $y = 2 \sin (x^2 + 2x - 1)$

1.13 $y = \cot^2 (1 + 3x^2)$

1.14 $y = \sin \dfrac{1}{x}$

1.15 $y = x \sin \dfrac{1}{x}$

1.16 $y = (2 + \sec^2 x)^3$

1.17 $y = \sin (\cos x)$

1.18 $y = \sqrt[4]{\sec(\tan x)}$

1.19 $f(x) = (1 + \cos^3(x^2))^2$

1.20 $f(x) = [x \sin x - x^2 \cos 3x]^{12}$

1.21 $f(x) = \tan \left(\dfrac{x+1}{x-1}\right)$

1.22 $h(x) = (\tan 2x)^3 \sin(1-x^2)$

1.23 $f(x) = \sin (\cos(\sin^2 x + \tan (\sin x)))$

1.24 $f(x) = \cos [x^3 + \sin (x^2-x)]^2$

1.25 $y = \left[\dfrac{\cos(2t - 1)}{\cot(t^2 + 1)}\right]^3$

1.26 $y = \sqrt{a^2 \sin(x/a)}$

1.27 $y = \sin \left(\cos \sqrt{x^2 + 1}\right)$

1.28 $y = \sin (\cos(x^2 + \tan x))$

1.29 $y = \left(1 + \sqrt{1 + \cos^3 x}\right)^{2001}$

1.30 $y = |\cos x|$

1.31 $y = \sin |x|$

1.32 $y = \tan |\sec x|$

2. จงหาอนุพันธ์ของฟังก์ชันตรีโกณมิติผกผัน ในข้อต่อไปนี้

2.1 $y = \tan^{-1}x + \cot^{-1}x$

2.2 $y = \tan^{-1} \left(\dfrac{1}{x}\right)$

2.3 $y = \sec^{-1} \sqrt{x}$

2.4 $y = \tan^{-1} \left(\dfrac{2x - 1}{2x}\right)$

2.5 $y = \sin^{-1} \left(\dfrac{x}{a}\right) + \dfrac{\sqrt{a^2 + x^2}}{x}$

2.6 $y = x \cot^{-1} (1 + x^2)$

2.7 $y = (1 + \tan^{-1} x)^2$

2.8 $y = \sin (\tan^{-1} x)$

2.9 $y = \tan^{-1} \sqrt{x}$

2.10 $y = \sin^{-1} (\cos x)$

3. จงหาอนุพันธ์ของฟังก์ชันไม่เด่นชัดในข้อต่อไปนี้

3.1 $x + xy + \sin(2x + 3y) = 0$

3.2 $\cos(x+y) = y^2 \sin x$

3.3 $y = \sin(x+y) + \cos(x-y)$

3.4 $\tan^2(xy^3 + y) = x$

3.5 $\dfrac{xy^2}{1 + \sec y} = 1 + y^3$

3.6 $\sin^{-1} y + \cos^{-1} x = y$

3.7 $x = \cos^5 y + \cos y$

3.8 $\tan^{-1} y = 3x + y$

3.9 $\tan(xy) = x$

3.10 $y = x + \sin(xy)$

3.11 $x + \tan|y| = 1$

3.12 $y = \left[\sin^{-1}(2x^3)\right]^4 + 3\sqrt{\cot^{-1} 2y}$

4. จงหาอนุพันธ์ตามที่กำหนดในแต่ละข้อ

4.1 $\dfrac{d^2}{dx^2}(\cos x^5)$

4.2 $\dfrac{d}{dx} f(\sin x)$

4.3 $\dfrac{d}{dx} f(\tan x)$

4.4 $\dfrac{d^3}{dx^3} \sin^3 x$

4.5 $\dfrac{d^2}{dx^2} f(\cos x)$

4.6 $\dfrac{d^2}{dx^2} f(\sec x)$

5. จงแสดงว่า $\dfrac{d}{dx} \tan^{-1}(\cot x) = -1$

6. กำหนดให้ $g(x) = \cos^{-1}(\cos x)$ จงแสดงว่า $g'(x) = \dfrac{\sin x}{|\sin x|}$

7. กำหนดให้ $f'(0) = 1$ และ $f''(0) = -2$ จงหาค่าของ $\dfrac{d^2}{dx^2} f(\cos x)$ ณ จุด ซึ่ง $x = \dfrac{\pi}{2}$

8. จงหาจุดบนเส้นโค้ง $y = \cos^2 x + \sin x$ ในช่วง $[0, \pi]$ ที่ทำให้เส้นสัมผัสเส้นโค้งที่จุดดังกล่าวขนานกับแกน x

9. กำหนดให้ $f(x) = \sin(2x + 1)$ และ $g(x) = x^3 + 3$ จงหา $\dfrac{d}{dx} f(g(x))$

10. จงหาสมการของเส้นสัมผัสเส้นโค้ง $\sin^{-1} y + y \cot x = x$ ที่จุด $(1, 0)$

11. จงหา $\dfrac{dy}{dx}$ เมื่อกำหนด x และ y ในรูปสมการอิงตัวแปรเสริมต่อไปนี้

$x = 2\csc\theta - 3\theta$ และ $y = 14\cot\theta + 2^5$

12. กำหนดให้ $f(x) = \sin x$ จงหา

12.1 $f^{(245)}(x)$

12.2 $f^{(n)}(x)$ เมื่อ n เป็นจำนวนเต็มบวก

13. กำหนดให้ $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$

จงแสดงว่า $f'(0)$ หาค่าได้ แต่กราฟของ $y = f'(x)$ ไม่ต่อเนื่องที่ $x = 0$

14. จงพิจารณาว่าเส้นสัมผัสเส้นโค้ง $y = \sin x$ ที่จุด $(\pi, 0)$ ทำมุมเท่าใดกับแกน x

15. จงพิจารณาว่ากราฟของ $y = \tan x$ ทำมุมเท่าใดกับแกน x ขณะที่กราฟตัดแกน x

เฉลยชุดฝึกหัด 10

1. 1.1 $y' = (x \sec x + 2 \cot x)' = x' \sec x + x(\sec x)' + 2(\cot x)'$

$= \sec x + x \sec x \tan x - 2 \csc^2 x$

1.2 $$y' = \frac{(\sin x - \cos x)(\sin x + \cos x)' - (\sin x + \cos x)(\sin x - \cos x)'}{(\sin x - \cos x)^2}$$

$$= \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= \frac{-2(\sin^2 x + \cos^2 x)}{(\sin x - \cos x)^2} = \frac{-2}{(\sin x - \cos x)^2}$$

1.3 $$y' = \frac{(1+x \cos x)(\csc x)' - (1+x \cos x)' \csc x}{(1+x \cos x)^2}$$

$$= \frac{(1+x \cos x)(- \csc x \cot x) - (\cos x - x \sin x) \csc x}{(1+x \cos x)^2}$$

1.4 $$y' = \frac{(\csc x + \cot x)(\csc x - \cot x)' - (\csc x + \cot x)'(\csc x - \cot x)}{(\csc x + \cot x)^2}$$

$$= \frac{(\csc x + \cot x)(- \csc x \cot x + \csc^2 x) - (- \csc x \cot - \csc^2 x)(\csc x - \cot x)}{(\csc x + \cot x)^2}$$

$$= \frac{\csc x(\csc^2 x - \cot^2 x) + \csc x(\csc^2 x - \cot^2 x)}{(\csc x + \cot x)^2} = \frac{2 \csc x(\csc x - \cot x)}{\csc x + \cot x}$$

1.5 $$\frac{dy}{dx} = 2 \tan x \frac{d}{dx} \tan x = 2 \tan x \sec^2 x$$

1.6 $$\frac{dy}{dx} = 3 \sec^2 x \frac{d}{dx} \sec x = 3 \sec^3 x \tan x$$

1.7 $$g'(t) = 2 \sin t \frac{d}{dt} \sin t - 2 \cos t \frac{d}{dt} \cos t = 2 \sin t \cos t + 2 \cos t \sin t$$

$= 4 \sin t \cos t = 2 \sin 2t$

1.8 $f'(t) = 2(\sin t + \cos t)(\sin t + \cos t)'$

$= 2(\sin t + \cos t)(\cos t - \sin t) = 2(\cos^2 t - \sin^2 t) = 2 \cos 2t$

1.9 $\frac{dy}{dx} = \frac{d}{dx}\sin 4x = \frac{d\sin(4x)}{d(4x)} \cdot \frac{d(4x)}{dx} = 4\cos 4x$

1.10 $\frac{dy}{dx} = \frac{d}{dx}\csc(x^3+1) = \frac{d\csc(x^3+1)}{d(x^3+1)} \cdot \frac{d(x^3+1)}{dx}$

$= -3x^2\csc(x^3+1)\cot(x^3+1)$

1.11 $\frac{dy}{dx} = \frac{2d\cos^2(x^2)}{dx} = 2\frac{d\cos^2(x^2)}{d(\cos(x^2))} \cdot \frac{d\cos(x^2)}{dx^2} \cdot \frac{dx^2}{dx}$

$= 4\cos(x^2)(-\sin(x^2))(2x) = -8x\cos(x^2)\sin(x^2) = -4x\sin(2x^2)$

1.12 $\frac{dy}{dx} = 2\frac{d\sin(x^2+2x-1)}{d(x^2+2x-1)} \cdot \frac{d(x^2+2x-1)}{dx}$

$= 2\cos(x^2+2x-1)(2x+2) = 4(x+1)\cos(x^2+2x-1)$

1.13 $\frac{dy}{dx} = \frac{d}{dx}\cot^2(1+3x^2) = \frac{d\cot^2(1+3x^2)}{d[\cot(1+3x^2)]} \cdot \frac{d\cot(1+3x^2)}{d(1+3x^2)} \cdot \frac{d(1+3x^2)}{dx}$

$= 2\cot(1+3x^2)[-\csc^2(1+3x^2)](6x) = -12x\cot(1+3x^2)\csc^2(1+3x^2)$

1.14 $\frac{dy}{dx} = \frac{d\sin(1/x)}{d(1/x)} \cdot \frac{d(1/x)}{dx} = \cos(\frac{1}{x})(-\frac{1}{x^2}) = -\frac{1}{x^2}\cos(\frac{1}{x})$

1.15 $y' = (x)'\sin\frac{1}{x} + x(\sin\frac{1}{x})' = \sin(\frac{1}{x}) - \frac{1}{x}\cos(\frac{1}{x})$

1.16 $y' = 3(2+\sec^2x)^2(2+\sec^2x)' = 3(2+\sec^2x)^2[2\sec x(\sec x)']$

$= 3(2+\sec^2x)^2(2\sec^2x\tan x) = 6\sec^2x\tan x(2+\sec^2x)^2$

1.17 $\frac{dy}{dx} = \frac{d}{dx}\sin(\cos x) = \frac{d\sin(\cos x)}{d(\cos x)} \cdot \frac{d(\cos x)}{dx} = -\sin x\cos(\cos x)$

1.18 $\frac{dy}{dx} = \frac{d\sqrt[4]{\sec(\tan x)}}{dx} = \frac{d\sqrt[4]{u}}{du} \cdot \frac{du}{dx}$ (เมื่อ $u = \sec(\tan x)$)

$= \frac{1}{4\sqrt[4]{u^3}} \cdot \frac{d\sec(\tan x)}{d(\tan x)} \cdot \frac{d\tan x}{dx} = \frac{\sec(\tan x)\tan(\tan x)\,.\sec^2x}{4\sqrt[4]{\sec^2(\tan x)}}$

1.19 $f'(x) = 2[1+\cos^3(x^2)][1+\cos^3(x^2)]' = 2(1+\cos^3(x^2))(3\cos^2(x^2))(\cos(x^2))'$

$= 6\cos^2(x^2)(1+\cos^3(x^2))(-\sin(x^2))(x^2)'$

$= -12x\ \sin(x^2)\ \cos^2(x^2)(1+\cos^3(x^2))$

1.20 $f'(x) = 12[x\sin x - x^2\cos 3x]^{11}[x\sin x - x^2\cos 3x]'$

$= 12(x\sin x - x^2\cos 3x)^{11}(\sin x + x\cos x - 2x\cos 3x + 3x^2\ \sin\ 3x)$

1.21 $f'(x) = \frac{d\tan u}{dx} = \frac{d\tan u}{du}\cdot\frac{du}{dx}$ (เมื่อ $u = \frac{x+1}{x-1}$)

$= \sec^2 u\cdot\left[\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}\right] = \frac{-2}{(x-1)^2}\sec^2(\frac{x+1}{x-1})$

1.22 $h'(x) = [(\tan 2x)^3]'\sin(1-x^2) + (\tan 2x)^3[\sin(1-x^2)]'$

$= 3(\tan 2x)^2(\tan 2x)'\sin(1-x^2) + (\tan 2x)^3\cos(1-x^2)(1-x^2)'$

$= 6\tan^2 2x\ \sec^2 2x\sin(1-x^2) - 2x\tan^3 2x\cos(1-x^2)$

1.23 ให้ $u = \sin x$, $v = u^2 + \tan u$ และ $w = \cos v$ แล้ว

$f'(x) = \frac{d}{dx}\sin(\cos(\sin^2 x + \tan(\sin x)))$

$= \frac{d}{dx}\sin w = \frac{d}{dw}\sin w\cdot\frac{dw}{dx} = \cos w\frac{d}{dv}\cos v\cdot\frac{dv}{dx}$

$= \cos w(-\sin v)\frac{d}{dx}\left[u^2+\tan u\right] = -\cos w.\sin v\left[2u\frac{du}{dx}+\frac{d}{du}\tan u\cdot\frac{du}{dx}\right]$

$= -\cos w\cdot\sin v[\sin 2x + \sec^2 u\cdot\cos x]$

$= -\cos(\cos(\sin^2 x + \tan(\sin x)))\cdot\sin(\sin^2 x + \tan(\sin x)[\sin 2x + \sec^2(\sin x)\cos x]$

1.24 ให้ $u = x^3 + \sin(x^2 - x)$ แล้ว $\frac{du}{dx} = 3x^2 + \cos(x^2 - x)(2x - 1)$ และดังนั้น

$f'(x) = \frac{d}{dx}\cos\left[x^3+\sin(x^2-x)\right]^2 = \frac{d}{dx}\cos u^2 = -\sin u^2(2u)\cdot\frac{du}{dx}$

$= -\sin[x^3+\sin(x^2-x)]^2\cdot 2(x^3+\sin(x^2-x))[3x^2+\cos(x^2-x)(2x-1)]$

1.25 ให้ $u = \dfrac{\cos(2t-1)}{\cot(t^2+1)}$ แล้วโดยกฎผลหาร จะได้

$$\frac{du}{dt} = \frac{\cot(t^2+1)[-\sin(2t-1)](2) - \cos(2t-1)[-\csc^2(t^2+1)](2t)}{\cot^2(t^2+1)}$$

$$= \frac{2t\cos(2t-1)\csc^2(t^2+1) - 2\sin(2t-1)\cot(t^2+1)}{\cot^2(t^2+1)}$$

$$\therefore \frac{dy}{dt} = \frac{du^3}{dt} = 3u^2 \cdot \frac{du}{dt}$$

$$= 3\left[\frac{\cos(2t-1)}{\cot(t^2+1)}\right]^2\left[\frac{2t\cos(2t-1)\csc^2(t^2+1) - 2\sin(2t-1)\cot(t^2+1)}{\cot(t^2+1)}\right]$$

1.26 ให้ $u = a^2\sin(x/a)$ แล้ว $\dfrac{du}{dx} = a^2\cos(x/a)\dfrac{d}{dx}(\dfrac{x}{a}) = a\cos(x/a)$

ดังนั้น $\dfrac{dy}{dx} = \dfrac{d\sqrt{u}}{dx} = \dfrac{d\sqrt{u}}{du}\cdot\dfrac{du}{dx} = \dfrac{1}{2\sqrt{u}}\cdot a\cos(x/a) = \dfrac{a\cos(x/a)}{2\sqrt{a^2\sin(x/a)}}$

1.27 $\dfrac{dy}{dx} = \dfrac{d}{dx}\sin(\cos\sqrt{x^2+1}) = \cos(\cos\sqrt{x^2+1})\cdot\dfrac{d}{dx}\cos\sqrt{x^2+1}$

$$= \cos(\cos\sqrt{x^2+1})(-\sin\sqrt{x^2+1})\cdot\frac{d}{dx}\sqrt{x^2+1}$$

$$= \frac{-x}{\sqrt{x^2+1}}\cos(\cos\sqrt{x^2+1})\sin\sqrt{x^2+1}$$

1.28 $\dfrac{dy}{dx} = \dfrac{d}{dx}\sin(\cos(x^2+\tan x)) = \cos(\cos(x^2+\tan x))\dfrac{d}{dx}\cos(x^2+\tan x)$

$$= \cos(\cos(x^2+\tan x))(-\sin(x^2+\tan x))\frac{d}{dx}(x^2+\tan x)$$

$$= -(2x+\sec^2 x)\sin(x^2+\tan x)\cos(\cos(x^2+\tan x))$$

1.29 $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(1+\sqrt{1+\cos^3 x}\right)^{2001} = 2001\left(1+\sqrt{1+\cos^3 x}\right)^{2000}\dfrac{d}{dx}\left(1+\sqrt{1+\cos^3 x}\right)$

$$= 2001\left(1+\sqrt{1+\cos^3 x}\right)^{2000}\cdot\frac{3\cos^2 x(-\sin x)}{2\sqrt{1+\cos^3 x}}$$

1.30 $\dfrac{dy}{dx} = \dfrac{\cos x}{|\cos x|}\cdot\dfrac{d}{dx}\cos x = \dfrac{-\cos x\sin x}{|\cos x|}$

1.31 $\dfrac{dy}{dx} = \dfrac{d}{dx}\sin|x| = \cos|x|\dfrac{d|x|}{dx} = \dfrac{x}{|x|}\cos|x|$

1.32 $\frac{dy}{dx} = \frac{d}{dx}\tan|\sec x| = \sec^2|\sec x|\frac{d}{dx}|\sec x| = \sec^2|\sec x| \cdot \frac{\sec x}{|\sec x|} \cdot \frac{d \sec x}{dx}$

$= \sec^2|\sec x| \cdot \frac{\sec x}{|\sec x|} \cdot \sec x \tan x = \sec^2|\sec x| \cdot \frac{\sec^2 x \tan x}{|\sec x|}$

2. 2.1 $\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1} x + \cot^{-1} x) = \frac{d}{dx}\tan^{-1} x + \frac{d}{dx}\cot^{-1} x = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$

2.2 $\frac{dy}{dx} = \frac{d}{dx}\tan^{-1}(1/x) = \frac{1}{1+(1/x)^2} \cdot \frac{d(1/x)}{dx} = \frac{x^2}{1+x^2} \cdot (\frac{-1}{x^2}) = -\frac{1}{1+x^2}$

2.3 $\frac{dy}{dx} = \frac{d}{dx}\sec^{-1}(\sqrt{x}) = \frac{d \sec^{-1}(\sqrt{x})}{d\sqrt{x}} \cdot \frac{d\sqrt{x}}{dx} = \frac{1}{\sqrt{x}\sqrt{x-1}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x\sqrt{x-1}}$

2.4 ให้ $u = \frac{2x-1}{2x}$ แล้ว $\frac{du}{dx} = \frac{2x(2) - (2x-1)(2)}{4x^2} = \frac{1}{2x^2}$ ดังนั้น

$\frac{dy}{dx} = \frac{d}{dx}\tan^{-1}(\frac{2x-1}{2x}) = \frac{d}{dx}\tan^{-1} u = \frac{d}{du}\tan^{-1} u \cdot \frac{du}{dx}$

$= \frac{1}{1+u^2} \cdot \frac{1}{2x^2} = \frac{1}{1+(\frac{2x-1}{2x})^2} \cdot \frac{1}{2x^2} = \frac{2}{8x^2 - 4x+1}$

2.5 $\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}(\frac{x}{a}) + \frac{d}{dx}\left(\frac{\sqrt{a^2+x^2}}{x}\right)$

$= \frac{1}{\sqrt{1-(x/a)^2}}\frac{d}{dx}(\frac{x}{a}) + (\frac{x(\sqrt{a^2+x^2})' - \sqrt{a^2+x^2}\,(x)'}{x^2})$

$= \frac{1}{\sqrt{1-x^2/a^2}} \cdot \frac{1}{a} + \frac{1}{x^2}\left[\frac{x}{2\sqrt{a^2+x^2}}(2x) - \sqrt{a^2+x^2}\right]$

$= \frac{1}{\sqrt{a^2-x^2}} + \frac{1}{\sqrt{a^2+x^2}} - \frac{\sqrt{a^2+x^2}}{x^2}$

2.6 $\frac{dy}{dx} = \cot^{-1}(1+x^2) + x \cdot \frac{-1}{1+(1+x^2)^2} \cdot (1+x^2)' = \cot^{-1}(1+x^2) - \frac{2x^2}{2+2x^2+x^4}$

2.7 $\frac{dy}{dx} = \frac{d}{dx}(1+\tan^{-1} x)^2 = 2(1+\tan^{-1} x)\frac{d}{dx}(1+\tan^{-1} x) = \frac{2(1+\tan^{-1} x)}{1+x^2}$

2.8 $\frac{dy}{dx} = \frac{d}{dx}\sin(\tan^{-1}x) = \cos(\tan^{-1}x) \cdot \frac{d}{dx}(\tan^{-1}x) = \frac{\cos(\tan^{-1}x)}{1+x^2}$

2.9 $\frac{dy}{dx} = \frac{d}{dx}\tan^{-1}\sqrt{x} = \frac{1}{1+(\sqrt{x})^2} \cdot \frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{x}(1+x)}$

2.10 $\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}(\cos x) = \frac{1}{\sqrt{1-\cos^2 x}} \cdot \frac{d(\cos x)}{dx}$

$= \frac{-\sin x}{\sqrt{1-\cos^2 x}} = \frac{-\sin x}{\sin x} = -1$

3. 3.1 $1 + y + xy' + \cos(2x + 3y)(2 + 3y') = 0$

หรือ $[x + 3\cos(2x + 3y)]y' = -(1 + y + 2\cos(2x + 3y))$

$\therefore\ y' = \frac{-(1 + y + 2\cos(2x + 3y))}{x + 3\cos(2x + 3y)}$

3.2 $-\sin(x + y)[1 + y'] = 2y\sin x \cdot y' + \cos x \cdot y^2$

หรือ $[2y\sin x + \sin(x + y)]y' = -[\sin(x + y) + y^2\cos x]$

$\therefore\ y' = \frac{-[\sin(x+y) + y^2\cos x]}{2y\sin x + \sin(x+y)}$

3.3 $y' = \cos(x+y)(1+y') - \sin(x-y)(1-y')$

หรือ $[\cos(x+y) + \sin(x-y) - 1]y' = \sin(x-y) - \cos(x+y)$

$\therefore\ y' = \frac{\sin(x-y) - \cos(x+y)}{\cos(x+y) + \sin(x-y) - 1}$

3.4 $2\tan(xy^3+y)\sec^2(xy^3+y)(y^3 + 3xy^2y' + y') = 1$

หรือ $2\tan(xy^3+y)\sec^2(xy^3+y)(3xy^2+1)y' = 1 - 2y^3\tan(xy^3+y)\sec^2(xy^3+y)$

$\therefore\ y' = \frac{1 - 2y^3\tan(xy^3+y)\sec^2(xy^3+y)}{2(3xy^2+1)\tan(xy^3+y)\sec^2(xy^3+y)}$

3.5 $\frac{(1+\sec y)(xy^2)' - (1+\sec y)'(xy^2)}{(1+\sec y)^2} = 3y^2 \cdot y'$

หรือ $(1+\sec y)(y^2 + 2xy \cdot y') - \sec y\tan y\,(xy^2)y' = (1+\sec y)^2 \cdot 3y^2y'$

หรือ $y^2(1+\sec y) = [3y^2(1+\sec y)^2 + xy^2\sec y\tan y - 2xy(1+\sec y)]y'$

$\therefore\ y' = \frac{y^2(1+\sec y)}{3y^2(1+\sec y)^2 + xy^2\sec y\tan y - 2xy(1+\sec y)}$

3.6 $\frac{y'}{\sqrt{1-y^2}} - \frac{1}{\sqrt{1-x^2}} = y'$ ทำให้ได้ $(\frac{1}{\sqrt{1-y^2}} - 1)y' = \frac{1}{\sqrt{1-x^2}}$

3.7 $1 = 5\cos^4 y\,(-\sin y)y' - \sin y \,.\, y' = -\sin y(5\cos^4 y + 1)y'$

$\therefore\ y' = -\frac{1}{\sin y\,(5\cos^4 y + 1)}$

3.8 $\frac{y'}{1+y^2} = 3 + y'$ หรือ $(\frac{1}{1+y^2} - 1)y' = 3 \quad \therefore\ y' = \frac{-3(1+y^2)}{y^2}$

3.9 $\sec^2(xy)(y + xy') = 1$ หรือ $x\sec^2(xy)y' = 1 - y\sec^2(xy)$

$\therefore\ y' = \frac{1 - y\sec^2(xy)}{x\sec^2(xy)}$

3.10 $y' = 1 + \cos(xy)(y + xy')$ หรือ $[1 - x\cos(xy)]y' = 1 + y\cos(xy)$

$y' = \frac{1 + y\cos(xy)}{1 - x\cos(xy)}$

3.11 $1 + \sec^2|y|\,(\frac{y.y'}{|y|}) = 0$ ทำให้ได้ $y' = \frac{-|y|}{y\sec^2|y|} = \frac{-|y|}{y}\cos^2|y|$

3.12 $y' = 4\left[\sin^{-1}(2x^3)\right]^3 \frac{d}{dx}\sin^{-1}(2x^3) + \frac{3}{2\sqrt{\cot^{-1} 2y}}\frac{d}{dx}\cot^{-1} 2y$

$= 4\left[\sin^{-1}(2x^3)\right]^3 \frac{6x^2}{\sqrt{1-4x^6}} - \frac{6y'}{2\sqrt{\cot^{-1} 2y}\,(1+4y^2)}$

หรือ $\left(1 + \frac{3}{\sqrt{\cot^{-1} 2y}\,(1+4y^2)}\right)y' = \frac{24x^2[\sin^{-1}(2x^3)]^3}{\sqrt{1-4x^6}}$

$\therefore\ y' = \frac{24x^2[\sin^{-1}(2x^3)]^3(1+4y^2)\sqrt{\cot^{-1} 2y}}{\sqrt{1-4x^6}\,[3 + (1+4y^2)\sqrt{\cot^{-1} 2y}]}$

4. 4.1 $\frac{d}{dx}(\cos x^5) = \frac{d}{dx^5}(\cos x^5)\,.\,\frac{dx^5}{dx} = (-\sin x^5)(5x^4) = -5x^4\sin(x^5)$

$\frac{d^2}{dx^2}(\cos x^5) = \frac{d}{dx}(-5x^4\sin(x^5)) = [\frac{d}{dx}(-5x^4)]\,.\,\sin(x^5) + (-5x^4)\frac{d}{dx}\sin(x^5)$

$= -20x^3\sin(x^5) - 5x^4\cos(x^5)\,5x^4$

$= -20x^3\sin(x^5) - 25x^8\cos(x^5)$

4.2 $\frac{d}{dx} f(\sin x) = \frac{d\, f(\sin x)}{d(\sin x)} . \frac{d}{dx} \sin x = f'(\sin x) . \cos x$

4.3 $\frac{d}{dx} f(\tan x) = \frac{df(\tan x)}{d \tan x} . \frac{d}{dx} \tan x = f'(\tan x) . \sec^2 x$

4.4 $\frac{d}{dx} \sin^3 x = \frac{d \sin^3 x}{d(\sin x)} . \frac{d}{dx} \sin x = 3 \sin^2 x \cos x$

$= 3(1 - \cos^2 x) \cos x = 3 \cos x - 3 \cos^3 x$

$\frac{d^2}{dx^2} \sin^3 x = \frac{d}{dx} (3 \cos x - 3 \cos^3 x) = - 3 \sin x - (3)(3) \cos^2 x \frac{d}{dx} \cos x$

$= - 3 \sin x + 9 \cos^2 x \sin x = - 3 \sin x + 9(1 - \sin^2 x) \sin x$

$= - 3 \sin x + 9 \sin x - 9 \sin^3 x = 6 \sin x - 9 \sin^3 x$

$\frac{d^3}{dx^3} \sin^3 x = \frac{d}{dx} (6 \sin x - 9 \sin^3 x) = 6 \cos x - 9(3 \sin^2 x) \cos x$

$= 6 \cos x - 9(3 \cos x - 3 \cos^3 x) = 27 \cos^3 x - 21 \cos x$

4.5 $\frac{d}{dx} f(\cos x) = f'(\cos x) \frac{d}{dx} \cos x = - \sin x\, f'(\cos x)$

$\frac{d^2}{dx^2} f(\cos x) = \frac{d}{dx} (- \sin x\, f'(\cos x)) = - \cos x\, f'(\cos x) - \sin x\, f''(\cos x)(\cos x)'$

$= - \cos x\, f'(\cos x) + \sin^2 x\, f''(\cos x)$

4.6 $\frac{d}{dx} f(\sec x) = f'(\sec x)(\sec x)' = \sec x \tan x\, f'(\sec x)$

$\frac{d^2}{dx^2} f(\sec x) = \frac{d}{dx} \sec x \tan x\, f'(\sec x)$

$= (\sec x)' \tan x\, f'(\sec x) + \sec x\, (\tan x)' f'(\sec x) + \sec x \tan x\, (f'(\sec x))'$

$= \sec x \tan^2 x\, f'(\sec x) + \sec^3 x\, f'(\sec x) + \sec^2 x \tan^2 x\, f''(\sec x)$

5. ให้ $u = \cot x$ แล้ว $\frac{du}{dx} = - \csc^2 x$ และดังนั้น

$\frac{d}{dx} \tan^{-1}(\cot x) = \frac{d}{dx} \tan^{-1} u = \frac{d}{du} \tan^{-1} u . \frac{du}{dx} = \frac{- \csc^2 x}{1+u^2}$

$= \frac{- \csc^2 x}{1 + \cot^2 x} = \left(\frac{-1}{\sin^2 x}\right)\left(\frac{\sin^2 x}{\sin^2 x + \cos^2 x}\right) = -1/1 = -1$

6. $g'(x) = \frac{d}{dx} \cos^{-1} (\cos x) = \frac{d \cos^{-1} (\cos x)}{d(\cos x)} . \frac{d}{dx} \cos x$

$= - \frac{1}{\sqrt{1 - \cos^2 x}} . (- \sin x) = \frac{\sin x}{\sqrt{\sin^2 x}} = \frac{\sin x}{|\sin x|}$

7. จากข้อ 4.5 เราจะได้

$$\left[\frac{d^2}{dx^2} f(\cos x)\right]_{x=\pi/2} = -\cos\frac{\pi}{2} f'(\cos\frac{\pi}{2}) + \sin^2(\frac{\pi}{2}) \cdot f''(\cos\frac{\pi}{2})$$

$$= -(0) \cdot f'(0) + (1) f''(0) = 0 + (1)(-2) = -2$$

8. $y' = \frac{d}{dx}(\cos^2 x + \sin x) = \frac{d}{dx}\cos^2 x + \frac{d}{dx}\sin x = -2\cos x \sin x + \cos x = 0$

$\Leftrightarrow \cos x(1 - 2\sin x) = 0 \Leftrightarrow \cos x = 0$ หรือ $1 - 2\sin x = 0,\ x \in [0, \pi]$

$\Leftrightarrow x = \frac{\pi}{2}$ หรือ $x = \frac{\pi}{6}$ หรือ $x = \frac{5\pi}{6}$

เมื่อ $x = \frac{\pi}{2}$ จะได้ $y = \cos^2(\frac{\pi}{2}) + \sin\frac{\pi}{2} = 0 + 1 = 1$

เมื่อ $x = \frac{\pi}{6}$ จะได้ $y = \cos^2(\frac{\pi}{6}) + \sin(\frac{\pi}{6}) = (\frac{\sqrt{3}}{2})^2 + \frac{1}{2} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$

เมื่อ $x = \frac{5\pi}{6}$ จะได้ $y = \cos^2(\frac{5\pi}{6}) + \sin(\frac{5\pi}{6}) = (\frac{-\sqrt{3}}{2})^2 + \frac{1}{2} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$

เพราะฉะนั้นจุดที่ต้องการคือ $(\frac{\pi}{2}, 1)$, $(\frac{\pi}{6}, \frac{5}{4})$ และ $(\frac{5\pi}{6}, \frac{5}{4})$

9. จาก $f(x) = \sin(2x+1)$ และ $g(x) = x^2+3$ ทำให้ได้

$f'(x) = 2\cos(2x+1)$ และ $g'(x) = 3x^2$

และได้

$$f'(g(x)) = 2\cos[2g(x)+1] = 2\cos[2(x^3+3)+1] = 2\cos(2x^3+7)$$

เพราะฉะนั้น

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x) = 2\cos(2x^3+7) \cdot (3x^2) = 6x^2\cos(2x^3+7)$$

10. $\left(\frac{d \sin^{-1} y}{dy}\right) y' + y' \cot x + y(-\csc^2 x) = 1$ หรือ $\frac{y'}{\sqrt{1-y^2}} + y' \cot x = 1 + y \csc^2 x$

ดังนั้นที่จุด (1, 0) จะได้ความสัมพันธ์ $y'(1 + \cot 1) = 1$ หรือ $y' = \frac{1}{1 + \cot 1}$

เพราะฉะนั้น สมการเส้นสัมผัสเส้นโค้งที่จุด (1, 0) คือ

$y - 0 = \frac{(x-1)}{1 + \cot 1}$ หรือ $(1 + \cot 1)y = x$

11. จาก $y = 14\cot\theta + 2^5$ และ $x = 2\csc\theta - 3\theta$

เราจะได้ $\frac{dy}{d\theta} = -14\csc^2\theta$ และ $1 = -2\csc\theta\cot\theta\frac{d\theta}{dx} - 3\frac{d\theta}{dx}$

ทำให้ได้ $\frac{d\theta}{dx} = \frac{-1}{3+2\csc\theta\cot\theta}$

เพราะฉะนั้น $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = (-14\csc^2\theta) \cdot (\frac{-1}{3+2\csc\theta\cot\theta}) = \frac{14\csc^2\theta}{3+2\csc\theta\cot\theta}$

12. $f(x) = \sin x \quad , \quad f'(x) = \cos x$

$f''(x) = -\sin x \quad , \quad f^{(3)}(x) = -\cos x$

$f^{(4)}(x) = \sin x \quad , \quad f^{(5)}(x) = \cos x$

$f^{(6)}(x) = -\sin x \quad , \quad f^{(7)}(x) = -\cos x$

$f^{(2m)}(x) = (-1)^m \sin x \quad , \quad f^{(2m+1)}(x) = (-1)^m \cos x$

12.1 จาก $245 = 244+1 = 2(122) + 1$ ดังนั้น $f^{(245)}(x) = (-1)^{122}\cos x = \cos x$

12.2 $$f^{(n)}(x) = \begin{cases} (-1)^m \sin x & \text{เมื่อ } n = 2m \\ (-1)^m \cos x & \text{เมื่อ } n = 2m+1 \end{cases}$$ เมื่อ m เป็นจำนวนเต็มบวก

13. $f'(0) = \lim\limits_{h\to 0}\dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h\to 0}\dfrac{f(h) - f(0)}{h} = \lim\limits_{h\to 0}\dfrac{h^2 \sin 1/h}{h} = \lim\limits_{h\to 0} h \sin\dfrac{1}{h}$

แต่ $-1 \leqslant \sin\frac{1}{h} \leqslant 1$ ทำให้ได้ $-h \leqslant h\sin\frac{1}{h} \leqslant h$ (เมื่อ $h \neq 0$)

ดังนั้นโดย squeezing theorem และ $\lim\limits_{h\to 0}(-h) = \lim\limits_{h\to 0}(h) = 0$ เราจะได้

$f'(0) = \lim\limits_{h\to 0} h \sin\dfrac{1}{h} = 0$

แต่ $$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

ซึ่ง $\lim\limits_{x\to 0} f'(x)$ หาค่าไม่ได้ เพราะฉะนั้น $y = f'(x)$ ไม่ต่อเนื่องที่ $x = 0$

14. สมมติให้เส้นสัมผัสเส้นโค้ง $y = \sin x$ ที่จุด $(\pi, 0)$ ทำมุม θ กับแกน x ดังนั้น

$\tan\theta$ = ความชันของเส้นสัมผัส = ความชันของเส้นโค้ง $y = \sin x$ ณ จุด $(\pi, 0)$

$= \left(\dfrac{d \sin x}{dx}\right)_{x=\pi} = (\cos x)_{x=\pi} = \cos\pi = -1$

ทำให้ได้ $\theta = \tan^{-1}(-1) = \dfrac{3\pi}{4}$

15. ขณะเส้นโค้ง $y = \tan x$ ตัดแกน x จะทำมุมกับแกน x เท่ากับมุมที่เส้นสัมผัสเส้นโค้งทำกับแกน x ณ จุด ที่ $y = \tan x$ ตัดแกน x และเนื่องจากกราฟ $y = \tan x$ ตัดแกน x ณ จุด $(n\pi, 0)$ เมื่อ n เป็นจำนวนเต็ม ดังนั้นโดยการพิจารณาเช่นเดียวกับข้อ 14 เมื่อ θ เป็นมุมที่ต้องการ จะได้

$\tan\theta = \dfrac{d \tan x}{dx}\Big|_{x=n\pi} = (\sec^2 x)_{x=n\pi} = 1$

เพราะฉะนั้น $\theta = \tan^{-1}(1) = \pi/4$