ชุดฝึกหัด 12

อนุพันธ์ของฟังก์ชันไฮเปอร์โบลิกและฟังก์ชันไฮเปอร์โบลิกผกผัน

จงหาอนุพันธ์ $\frac{dy}{dx}$ เมื่อกำหนด y ในข้อ 1–30 ต่อไปนี้

1. $y = e^x(\cosh x + \sinh x)$
2. $y = \tanh \frac{x}{2} + \frac{2x}{4+x^2}$
3. $y = \cosh (2x^3-1)$
4. $y = \sqrt{\tanh x}$
5. $y = \ln (\sinh^2 x)$
6. $y = \operatorname{csch}^2 (x^2+1)$
7. $y = x \operatorname{sech} x^2$
8. $y = \ln \sqrt{\tanh 2x}$
9. $y = \dfrac{\coth (5x)^{5/2} (\tanh 3x)^{1/3}}{3x + 4 \cos x}$
10. $y = \ln \sqrt{1-x^2} + x \tanh^{-1} x$
11. $y = \sinh^{-1} \frac{x}{2}$
12. $y = \cosh^{-1} \frac{1}{x}$
13. $y = \tan^{-1} (\sinh x)$
14. $y = \dfrac{\coth^{-1} x}{x}$
15. $y = \sqrt{1+x} + \operatorname{csch}^{-1} \sqrt{x}$
16. $y = x^2 \cosh^{-1} x$
17. $y = \coth^{-1} \frac{1}{x}$
18. $y = \cosh^{-1} e^x$
19. $y = \tanh^{-1} (x^2-1)$
20. $y = (\sinh^{-1} x + \sin^{-1} x)^2$
21. $y = \tanh^{-1} (\cos 2x)$
22. $y = \sinh^{-1} (\sin x)$
23. $y = \tanh^{-1} (\tan x)$
24. $y = \ln (\cosh^3 (2^x))$
25. $y = 2^{ex} \cdot x^{\pi x^4} \tanh^{-1} x$
26. $y = \tanh [\sec(x^4+4x^3-6x^2)^3]$
27. $y = \tanh^{-1} (\sin x)$
28. $y = 2 \tanh^{-1} (\tan \frac{x}{2})$
29. $y = \operatorname{sech}^{-1} (\cos x)$
30. $x = a \operatorname{sech}^{-1} \frac{y}{a} - \sqrt{a^2 - y^2}$

31. จงหาอนุพันธ์อันดับสองของฟังก์ชันที่กำหนดในข้อต่อไปนี้

    31.1 $y = \tanh x$

    31.2 $y = \sinh x$

    31.3 $y = a \cosh \frac{x}{a}$

32. จงหาค่าของลิมิตในข้อต่อไปนี้

    32.1 $\lim_{x\to 0} \dfrac{\sinh x}{x}$

    32.2 $\lim_{x\to 0} \dfrac{\cosh x - 1}{x}$

เฉลยชุดฝึกหัด 12

1. $\frac{dy}{dx} = (\frac{de^x}{dx})(\cosh x + \sinh x) + e^x(\frac{d}{dx}\cosh x + \frac{d}{dx}\sinh x)$

$= e^x(\cosh x + \sinh x) + e^x(\sinh x + \cosh x) = 2e^x(\cosh x + \sinh x)$

2. $\frac{dy}{dx} = \frac{d}{dx}(\tanh \frac{x}{2}) + \frac{d}{dx}(\frac{2x}{4+x^2}) = (\text{sech}^2 \frac{x}{2}) \frac{d}{dx}(\frac{x}{2}) + \frac{(4+x^2)(2) - (2x)(2x)}{(4+x^2)^2}$

$= \frac{1}{2}\text{sech}^2 \frac{x}{2} + \frac{8+2x^2 - 4x^2}{(4+x^2)^2} = \frac{1}{2}\text{sech}^2 \frac{x}{2} + \frac{2(4-x^2)}{(4+x^2)^2}$

3. $\frac{dy}{dx} = \frac{d}{dx}\cosh(2x^3 - 1) = \sinh(2x^3 - 1)\frac{d}{dx}(2x^3 - 1) = 6x^2\sinh(2x^3 - 1)$

4. $\frac{dy}{dx} = \frac{d\sqrt{\tanh x}}{dx} = \frac{1}{2\sqrt{\tanh x}}\frac{d(\tanh x)}{dx} = \frac{\text{sech}^2 x}{2\sqrt{\tanh x}}$

5. $\frac{dy}{dx} = \frac{d}{dx}\ln(\sinh^2 x) = \frac{1}{\sinh^2 x}\frac{d}{dx}(\sinh^2 x) = \frac{2\sinh x \cdot \cosh x}{\sinh^2 x} = 2\coth x$

6. $\frac{dy}{dx} = \frac{d}{dx}\text{csch}^2(x^2+1) = 2\,\text{csch}(x^2+1)\frac{d}{dx}\text{csch}(x^2+1)$

$= -4x\,\text{csch}^2(x^2+1)\coth(x^2+1)$

7. $\frac{dy}{dx} = \text{sech}\, x^2 + x(-\text{sech}\, x^2 \tanh x^2)(2x) = \text{sech}\, x^2(1 - 2x^2\,\text{sech}\, x^2 \tanh x^2)$

8. $\frac{dy}{dx} = \frac{d}{dx}\ln\sqrt{\tanh 2x} = \frac{1}{\sqrt{\tanh 2x}}\frac{d}{dx}\sqrt{\tanh 2x}$

$= \frac{1}{\sqrt{\tanh 2x}} \cdot \frac{1}{2\sqrt{\tanh 2x}}\frac{d}{dx}(\tanh 2x)$

$= \frac{1}{2\tanh 2x} \cdot \text{sech}^2 2x \frac{d2x}{dx} = \frac{2\,\text{sech}^2 2x}{2\tanh 2x} = \frac{2}{\sinh 4x}$

9. ถ้า $y = \dfrac{(\coth 5x)^{5/2} (\tanh 3x)^{1/3}}{3x + 4\cos x}$ แล้ว

$\ln y = \dfrac{5}{2}\ln(\coth 5x) + \dfrac{1}{3}\ln(\tanh 3x) - \ln(3x + 4\cos x)$

ซึ่งทำให้ได้ $\dfrac{1}{y} \cdot y' = \dfrac{5(-\operatorname{csch}^2 5x)(5)}{2\coth 5x} + \dfrac{(\operatorname{sech}^2 3x)(3)}{3\tanh 3x} - \dfrac{(3 - 4\sin x)}{3x + 4\cos x}$

$= \dfrac{-25\operatorname{csch} 10x}{2} + \dfrac{2\operatorname{csch} 6x}{3} + \dfrac{4\sin x - 3}{3x + 4\cos x}$

$\therefore\ y' = \dfrac{(\coth 5x)^{5/2}(\tanh 3x)^{1/3}}{3x + 4\cos x}\left[\dfrac{2\operatorname{csch} 6x}{3} - \dfrac{25\operatorname{csch} 10x}{2} + \dfrac{4\sin x - 3}{3x + 4\cos x}\right]$

10. $\dfrac{dy}{dx} = \dfrac{d}{dx}[\ln\sqrt{1-x^2} + x\tanh^{-1} x] = \dfrac{d}{dx}\ln\sqrt{1-x^2} + \dfrac{d}{dx}x\tanh^{-1} x$

$= \dfrac{1}{2}\dfrac{d}{dx}\ln(1-x^2) + [\tanh^{-1} x + x\dfrac{d}{dx}\tanh^{-1} x]$

$= \dfrac{-2x}{2(1-x^2)} + \left[\tanh^{-1} x + \dfrac{x}{1-x^2}\right] = \tanh^{-1} x \ , \ |x| < 1$

11. $\dfrac{dy}{dx} = \dfrac{d}{dx}\sinh^{-1}(\dfrac{x}{2}) = \dfrac{1}{\sqrt{1 + (x/2)^2}}\dfrac{d}{dx}(\dfrac{x}{2}) = \dfrac{1}{2}\cdot\dfrac{2}{\sqrt{4+x^2}} = \dfrac{1}{\sqrt{4+x^2}}$

12. $\dfrac{dy}{dx} = \dfrac{d}{dx}\cosh^{-1}(\dfrac{1}{x}) = \dfrac{1}{\sqrt{(1/x)^2 - 1}}\cdot\dfrac{d}{dx}(\dfrac{1}{x}) \ , \ \dfrac{1}{x} > 1$

$= (-\dfrac{1}{x^2})\cdot\dfrac{x}{\sqrt{1-x^2}} = -\dfrac{1}{x\sqrt{1-x^2}} \ , \ x < 1$

13. $\dfrac{dy}{dx} = \dfrac{d}{dx}\tan^{-1}(\sinh x) = \dfrac{1}{1 + (\sinh^2 x)}\dfrac{d}{dx}\sinh x = \dfrac{\cosh x}{1 + \sinh^2 x} = \operatorname{sech} x$

14. $\dfrac{dy}{dx} = \dfrac{d}{dx}(\dfrac{\coth^{-1} x}{x}) = \dfrac{x(\coth^{-1} x)' - \coth^{-1} x}{x^2} = \dfrac{1}{x(1 - x^2)} - \dfrac{\coth^{-1} x}{x^2} \ , \ |x| > 1$

15. $\dfrac{dy}{dx} = \dfrac{d}{dx}[\sqrt{1+x} + \operatorname{csch}^{-1}\sqrt{x}] = \dfrac{d}{dx}\sqrt{1+x} + \dfrac{d}{dx}\operatorname{csch}^{-1}\sqrt{x}$

$= \dfrac{1}{2\sqrt{1+x}} + \dfrac{(-1)}{|\sqrt{x}|\sqrt{1+x}}\cdot\dfrac{d\sqrt{x}}{dx} \ , \ \sqrt{x} \neq 0$

$= \dfrac{1}{2\sqrt{1+x}} - \dfrac{1}{\sqrt{x(1+x)}}\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{x-1}{2x\sqrt{1+x}} \ , \ x \neq 0$

16. $\frac{dy}{dx} = \frac{d}{dx}(x^2 \cosh^{-1} x) = (\frac{dx^2}{dx}) \cosh^{-1} x + x^2 \frac{d}{dx} \cosh^{-1} x$

$= 2x \cosh^{-1} x + \frac{-x^2}{\sqrt{x^2-1}}$, $x \neq 0$

17. $\frac{dy}{dx} = \frac{d}{dx}(\coth^{-1} \frac{1}{x}) = \frac{1}{1 - (1/x)^2} \cdot \frac{d}{dx}(\frac{1}{x})$, $(\frac{1}{x^2}) > 1$

$= \frac{x^2}{x^2-1} \cdot (\frac{-1}{x^2}) = \frac{-1}{x^2-1}$, $x^2 < 1$

18. $\frac{dy}{dx} = \frac{d}{dx}(\cosh^{-1} e^x) = \frac{1}{\sqrt{e^{2x}-1}} \cdot \frac{de^x}{dx} = \frac{e^x}{\sqrt{e^{2x}-1}}$

19. $\frac{dy}{dx} = \frac{d}{dx} \tanh^{-1} (x^2-1) = \frac{1}{1 - (x^2-1)^2} \cdot \frac{d}{dx}(x^2-1)$

$= \frac{2x}{1-x^4+2x^2-1} = \frac{2x}{x^2 (2-x^2)} = \frac{2}{x(2-x^2)}$, $|x^2-1| < 1$

20. $\frac{dy}{dx} = \frac{d}{dx}(\sinh^{-1} x + \sin^{-1} x)^2 = 2(\sinh^{-1} x + \sin^{-1} x) \frac{d}{dx} (\sinh^{-1} x + \sin^{-1} x)$

$= 2(\sinh^{-1} x + \sin^{-1} x)\left(\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1-x^2}}\right)$

21. $\frac{dy}{dx} = \frac{d}{dx} \tanh^{-1} (\cos 2x) = \frac{1}{1 - \cos^2 2x} \cdot \frac{d}{dx} \cos 2x$

$= \frac{-2 \sin 2x}{1 - \cos^2 2x} = \frac{-2 \sin 2x}{\sin^2 2x} = -2 \csc 2x$

22. $\frac{dy}{dx} = \frac{d}{dx} \sinh^{-1} (\sin x) = \frac{1}{\sqrt{1 + \sin^2 x}} \frac{d \sin x}{dx} = \frac{\cos x}{\sqrt{1 + \sin^2 x}}$

23. $\frac{dy}{dx} = \frac{d}{dx} \tanh^{-1} (\tan x) = \frac{1}{1 - \tan^2 x} \cdot \frac{d}{dx} \tan x = \frac{\sec^2 x}{1 - \tan^2 x} = \sec 2x$

24. $\frac{dy}{dx} = \frac{d}{dx} \ln(\cosh^3(2^x)) = \frac{1}{\cosh^3(2^x)} \cdot \frac{d}{dx} \cosh^3(2^x)$

$= \frac{3 \cosh^2(2^x)}{\cosh^3(2^x)} \cdot \frac{d}{dx} \cosh(2^x) = \frac{3 \sinh(2^x)}{\cosh(2^x)} \cdot \frac{d2^x}{dx} = 3.2^x \tanh(2^x)$

25. $\frac{d2^{ex}}{dx} = 2^{ex} \ln 2 \frac{de^x}{dx} = e^x \cdot 2^{ex} \cdot \ln 2$

ให้ $u = x^{\pi x^4}$ แล้ว $\ln u = \pi x^4 \ln x$ ทำให้ได้ $\frac{1}{u} \cdot u' = 4\pi x^3 \ln x + \frac{\pi x^4}{x}$

และดังนั้น $u' = x^{\pi x^4}(4\pi x^3 \ln x + \pi x^3) = \pi x^3 x^{\pi x^4}(4 \ln x + 1)$

เพราะฉะนั้น

$$\frac{dy}{dx} = \frac{d}{dx} 2^{ex} \quad x^{\pi x^4} \tanh^{-1} x$$

$$= (2^{ex})' \cdot x^{\pi x^4} \cdot \tanh^{-1} x + 2^{ex} (x^{\pi x^4})' \tanh^{-1} x + 2^{ex} \cdot x^{\pi x^4} (\tanh^{-1} x)'$$

$$= e^x 2^{ex} \ln 2 \, x^{\pi x^4} \tanh^{-1} x + 2^{ex} x^{\pi x^4} \pi x^3 (4 \ln x + 1) \tanh^{-1} x + \frac{2^{ex} x^{\pi x^4}}{1-x^2}$$

$$= 2^{ex} x^{\pi x^4} (e^x \ln 2 \tanh^{-1} x + \pi x^3 (4 \ln x + 1) \tanh^{-1} x + \frac{1}{1-x^2})$$

26. $\frac{dy}{dx} = \frac{d}{dx} \tanh [\sec (x^4+4x^3-6x^2)^3] = \operatorname{sech}^2 [\sec (x^4+4x^3-6x^2)^3] \frac{d}{dx} \sec (x^4+4x^3-6x^2)^3$

$$= \operatorname{sech}^2 [\sec (x^4+4x^3-6x^2)^3] \sec (x^4+4x^3-6x^2)^3 \tan (x^4+4x^3-6x^2)^3 \frac{d}{dx} (x^4+4x^3-6x^2)^3$$

$$= 3(x^4+4x^3-6x^2)^2(4x^3+12x^3-12x)\operatorname{sech}^2 [\sec (x^4+4x^3-6x^2)^3] \sec (x^4+4x^3-6x^2)^3 \tan (x^4+4x^3-6x^2)^3$$

27. $\frac{dy}{dx} = \frac{d}{dx} \tanh^{-1} (\sin x) = \frac{1}{1 - \sin^2 x} \cdot \frac{d \sin x}{dx} = \frac{\cos x}{\cos^2 x} = \sec x$

28. $\frac{dy}{dx} = \frac{d}{dx} 2 \tanh^{-1} (\tan \frac{x}{2}) = \frac{2}{1 - \tan^2 x/2} \cdot \frac{d}{dx} \tan(\frac{x}{2})$

$$= \frac{2 \sec^2 (x/2)}{1 - \tan^2 (x/2)} \cdot \frac{d}{dx} (\frac{x}{2}) = \frac{\sec^2 x/2}{1 - \tan^2 (x/2)} = \sec x$$

29. $\frac{dy}{dx} = \frac{d}{dx} \operatorname{sech}^{-1} (\cos x) = \frac{-1}{\cos x \sqrt{1-\cos^2 x}} \cdot \frac{d}{dx} (\cos x) = \frac{\sin x}{\cos x \sin x} = \sec x$

30. จาก $x = a \operatorname{sech}^{-1} \frac{y}{a} - \sqrt{a^2-y^2}$ เมื่อหาอนุพันธ์เทียบกับ x ตลอดสมการจะได้

$$1 = a \frac{d}{dx} \operatorname{sech}^{-1} \frac{y}{a} - \frac{d\sqrt{a^2-y^2}}{dx} = \frac{-a}{y/a \sqrt{1-(y/a)^2}} \cdot \frac{d}{dx} \frac{y}{a} - \frac{1}{2\sqrt{a^2-y^2}} \cdot \frac{d}{dx} (a^2-y^2)$$

$$= \frac{-a^2 y'}{y\sqrt{a^2-y^2}} + \frac{yy'}{\sqrt{a^2-y^2}} = \frac{(y^2-a^2)y'}{y\sqrt{a^2-y^2}}$$

$$\therefore y' = \frac{-y\sqrt{a^2-y^2}}{a^2-y^2} = \frac{-y}{\sqrt{a^2-y^2}}$$

31.

31.1 $y' = \frac{d}{dx}\tanh x = \text{sech}^2 x$

$y'' = \frac{d}{dx}\text{sech}^2 x = 2\,\text{sech}\,x \frac{d}{dx}\text{sech}\,x = -2\,\text{sech}^2 x \tanh x$

31.2 $y' = \frac{d}{dx}\sinh x = \cosh x$

$y'' = \frac{d}{dx}\cosh x = \sinh x$

31.3 $y' = \frac{d}{dx}(a \cosh \frac{x}{a}) = a(\sinh \frac{x}{a}) \cdot \frac{1}{a} = \sinh \frac{x}{a}$

$y'' = \frac{d}{dx}\sinh \frac{x}{a} = \cosh (\frac{x}{a}) \frac{d}{dx}(\frac{x}{a}) = \frac{1}{a} \cdot \cosh (\frac{x}{a})$

32.

32.1 $\lim_{x \to 0} \frac{\sinh x}{x} = \lim_{x \to 0} \frac{\sinh x - \sinh 0}{x-0} = (\sinh x)'|_{x=0} = \cosh 0 = 1$

32.2 $\lim_{x \to 0} \frac{\cosh x - 1}{x} = \lim_{x \to 0} \frac{\cosh x - \cosh 0}{x-0} = (\cosh x)'|_{x=0} = \sinh 0 = 0$